Operads of poset matrices

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Abstract

This paper establishes operad structures on the collection of poset matrices by introducing a new framework of partial composition operations. Extending the combinatorial setting of naturally labelled posets, we define several partial compositions that serve as basic tools for constructing poset matrices of arbitrary size. We prove that three of these operations satisfy the axioms of operads, thereby giving an operad structure to the set of poset matrices. We further characterize the dual operations and provide explicit combinatorial interpretations of these constructions in terms of naturally labelled posets. In addition, we address some open questions on poset enumeration via the operads of poset matrices.

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1 Introduction

Operads [12, 13] are algebraic structures designed to model the composition of operations and encode multilinear structures. In general, an operad formalizes the way in which several smaller structures of the same type can be composed to form a larger one, thus providing a powerful framework for both algebraic and combinatorial analysis. A central idea is that two operations x and y can be composed at the ith position by grafting the output of y into the input of x, yielding a new operation $x \circ_i y$. The notion of operads is particularly important and useful in categories with a good notion of homotopy, where they play a key role in organizing hierarchies of higher homotopies.

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Formally, a *nonsymmetric operad* (operad for short) is a collection $\mathcal{O} := \bigsqcup_{n \geqslant 1} \mathcal{O}(n)$ of objects equipped with partial composition maps

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n+m-1), \quad \text{for} \quad i \in [n] = \{1, \ldots, n\},$$

satisfying the following axioms for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $z \in \mathcal{O}(k)$:

(i) Nested associativity: $(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z)$, $i \in [n], j \in [m]$;

(ii) Parallel associativity: $(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y$, $\quad i < j \in [n]$;

(iii) Identity: $\mathrm{id} \circ_1 x = x \circ_i \mathrm{id} = x$, $i \in [n]$,

where $\mathrm{id} \in \mathcal{O}(1)$ is the identity element of $\mathcal{O}$.

Recent developments by Fauvet et al. [8] and Giraudo [9, 10] have explored operad structures arising from finite posets, focusing particularly on their representations via Hasse diagrams. These works highlight the foundational role of posets and operads in combinatorics and computer science–an interplay that has evolved over several decades. In this paper, we investigate operad structures on finite posets from the perspective of their incidence matrices, which we refer to as *poset matrices*. Although operads and posets have individually attracted considerable interest, the operadic formulation of poset matrices remains largely unexplored. Addressing this gap, we reinterpret the operadic frameworks developed in [8] within the matrix theoretic setting, thereby introducing new insights and directions for combinatorial and algebraic investigation.

A partially ordered set (*poset* for short) is an ordered pair $P = (X, \preceq)$ consisting of a set X and a partial order $\preceq$ on X that is reflexive, antisymmetric, and transitive. Let $X = [n] \subset \mathbb{N}$. A partial order $\preceq$ on $[n]$ is said to be *natural* if $x \preceq y$ implies $x \leqslant y$. We refer to such posets as *naturally labelled* (NL) posets (see [1]). By Szpilrajn's Theorem, every finite poset is isomorphic to an NL poset [7]. The incidence matrix $A = (a_{i,j})$ of an NL poset P on $[n]$ is defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } j \preceq i, \\ 0 & \text{otherwise.} \end{cases}$$

We call A the *poset matrix* of the poset P (also see [6, 14]). By definition, A is a poset matrix if and only if A is a $(0, 1)$-lower triangular matrix with 1s on the main diagonal and it is transitive *i.e.*, if $a_{i,j} = 1$ and $a_{j,k} = 1$ then $a_{i,k} = 1$. It is also known [4, 6] that A is an idempotent, Boolean matrix over the Boolean ring $\mathbb{B} = \{0, 1\}$ *i.e.*, $A = A^2$. There is a one-to-one correspondence between NL posets on $[n]$ and their associated poset matrices. Therefore, A is an $n \times n$ poset matrix if and only if the associated poset, denoted by P_A, is a naturally labeled poset on $[n]$. Two poset matrices A and B of order n are said to be *permutation equivalent* if there exists a permutation matrix Q such that $A = QBQ^T$. Hence, Birkhoff's question [2] about counting non-isomorphic posets on n elements is equivalent to counting poset matrices up to permutation equivalence.

The present work introduces a collection of partial composition maps on poset matrices, extending the classical operad framework. Among them, three are shown to satisfy the operad axioms. These structures not only yield a new operadic interpretation of NL posets but also lead to poset enumeration and classification problems in combinatorics. More specifically, this paper is organized as follows. Section 2 is devoted to the construction of operad structures using poset matrices and we present new structural properties resulting from these partial compositions. Section 3 investigates the duals of the operads of poset matrices. Section 4 translates the partial compositions on poset matrices into an explicit construction on NL posets. In Section 5, we pose some open questions on enumeration and forbidden subposets via operads of poset matrices. Finally, in Section 6, Appendix illustrates explicit operadic constructions of all non-isomorphic NL posets of size $n = 3, 4$.

2 Operad structure of poset matrices

We begin by introducing our main approach of constructing poset matrices by explicitly outlining the partial compositions $\circ_i$ defined on the combinatorial settings of poset matrices. Some of these partial compositions result to an operad structure on poset matrices. We denote by $\mathcal{PM}(n)$ the set of all $n \times n$ poset matrices and let

$$\mathcal{PM} = \bigsqcup_{n \geqslant 1} \mathcal{PM}(n).$$

For the construction of operads on the collection $\mathcal{PM}$ of poset matrices, we shall use some notations for an $n \times n$ matrix $A = [a_{i,j}]$. Let $\alpha, \beta \subset [n]$. Denote by $A[\alpha \mid \beta]$ the $|\alpha| \times |\beta|$ submatrix of A obtained by taking rows and columns indexed by all the elements of α and β, respectively. If $\alpha = \beta$, we simplify the notation to $A[\alpha]$.

Let $A = [a_{i,j}] \in \mathcal{PM}(n)$ be an $n \times n$ poset matrix. For a fixed index $i \in [n]$, define the index sets

$$\alpha_{i-1} = \{1, 2, \ldots, i-1\}, \quad \beta_{n-i} = \{i+1, i+2, \ldots, n\}, \tag{1}$$

where $\alpha_0 = \emptyset$ and $\beta_0 = \emptyset$. With respect to the partition of the index set $[n]$ into $(\alpha_{i-1}, \{i\}, \beta_{n-i})$, the matrix A admits the following block decomposition:

$$A = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{r}_i(A) & a_{i,i} & O \\ \hline A_{21} & \mathbf{c}_i(A) & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, 1 \\ \}\, n-i \end{array}. \tag{2}$$

In (2), the vectors

$$\mathbf{r}_i(A) := (a_{i,1}, \ldots, a_{i,i-1}) \quad \text{and} \quad \mathbf{c}_i(A) := (a_{i+1,i}, \ldots, a_{n,i})^T$$

are called the ith *sub-row vector* and the ith *sub-column vector* of A, respectively. If P denotes the NL poset associated with A, then for each $i \in P$, the vectors $\mathbf{r}_i(A)$ and $\mathbf{c}_i(A)$

encode the strict down-set $\downarrow i = \{j \in P \mid j \prec i\}$ and the strict up-set $\uparrow i = \{j \in P \mid i \prec j\}$, respectively.

The decomposition (2) will play a key role in the definition of the partial composition map $\circ_i$ on $\mathcal{PM}(n) \times \mathcal{PM}(m)$, obtained by replacing the diagonal entry $a_{i,i}$ by some poset matrix.

Operad structure via block substitution. Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. Our next goal is to define some partial compositions on $\mathcal{PM}$ such that for $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$, partial compositions on A and B yield matrices of the form (3):

$$A \circ_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline U_i & B & O \\ \hline A_{21} & V_i & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i \end{array}, \tag{3}$$

where U_i, V_i are (0,1)-matrices and A_{11}, A_{21} or A_{22} may be vacuous by virtue of having no rows or no columns. Clearly, $A \circ_i B$ is a lower triangular matrix with 1s on the main diagonal, and A_{11}, B, A_{22} are poset matrices.

In this work, we focus on the structures of U_i and V_i for which $A \circ_i B$ becomes a poset matrix for all $i \in [n]$. From now on, we denote by $O_{s,t}$ and $J_{s,t}$ the $s \times t$ zero matrix and all-ones matrix, respectively. When the size can be understood from the context we simply denote by O and J.

Theorem 1. *Let $A \in \mathcal{P}M(n)$ and $B \in \mathcal{P}M(m)$. For some $i \in [n]$, assume that $A \circ_i B$ in (3) is denoted by the following symbols according to each U_i, V_i, and A_{21} as follows:*

$$(a)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ J & B & O \\ J & J & A_{22} \end{pmatrix} \qquad (b)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ O & B & O \\ J & O & A_{22} \end{pmatrix}$$

$$(c)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ J & B & O \\ J & O & A_{22} \end{pmatrix} \qquad (d)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ O & B & O \\ J & J & A_{22} \end{pmatrix}$$

$$(e)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ O & B & O \\ O & O & A_{22} \end{pmatrix} \qquad (f)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ O & B & O \\ O & J & A_{22} \end{pmatrix}$$

$$(g)\ A \underset{i}{\boxplus} B = \begin{pmatrix} A_{11} & O & O \\ J & B & O \\ O & O & A_{22} \end{pmatrix}$$

Then $A \circ_i B$ is a poset matrix for each of the 7 cases (a)-(g).

Proof. For some $i \in [n]$, suppose that the submatrices U_i, V_i, and A_{21} in (3) can independently take one of the zero matrix O or the all-ones matrix J. It is clear that for each of the seven possible cases (a)-(g), $A \circ_i B$ preserves transitivity from the transitivity of both A and B. Thus $A \circ_i B$ is a poset matrix in $\mathcal{PM}(n + m - 1)$. □

Theorem 1 can be reformulated with two classical well-known operations on posets [15], *disjoint union* $+$ and *ordinal sum* $\oplus$, defined as follows. Let P and Q be posets. Then

(i) $x \preceq_{P+Q} y \iff x, y \in P$ and $x \preceq_P y$; or $x, y \in Q$ and $x \preceq_Q y$.

(ii) $x \preceq_{P\oplus Q} y \iff x, y \in P$ and $x \preceq_P y$; or $x, y \in Q$ and $x \preceq_Q y$; or $x \in P$ and $y \in Q$.

Hence $+$ is commutative but $\oplus$ is not. Take P and Q to be the NL posets associated with $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$, respectively. For each $i \in [n]$, denote by $P_{<i}$ the subposet of P formed by the elements of $[i-1]$ and by $P_{>i}$ the subposet of P formed by the elements of $[n] \setminus [i]$. Then, up to a natural reindexation, Theorem 1 can be reformulated as follows.

(a′) $P_{<i} \oplus Q \oplus P_{>i}$ (b′) $(P_{<i} \oplus P_{>i}) + Q$ (c′) $P_{<i} \oplus (Q + P_{>i})$ (d′) $(P_{<i} + Q) \oplus P_{>i}$;

(e′) $P_{<i} + Q + P_{>i}$ (f′) $P_{<i} + (Q \oplus P_{>i})$ (g′) $(P_{<i} \oplus Q) + P_{>i}$.

We also note that the eighth composition with $U_i = J$, $V_i = J$, $A_{21} = O$ is not possible since transitivity does not hold.

Let $A \in \mathcal{PM}(n)$. For $i \in [n]$, we define $\mathbf{R}_i^{\times m}(A)$ and $\mathbf{C}_i^{\times m}(A)$ to be the $m \times (i-1)$ matrix and the $(n-i) \times m$ matrix respectively as follows:

$$\mathbf{R}_i^{\times m}(A) := \underbrace{\begin{pmatrix} \mathbf{r}_i(A) \\ \vdots \\ \mathbf{r}_i(A) \end{pmatrix}}_{m \text{ identical rows}} \quad \text{and} \quad \mathbf{C}_i^{\times m}(A) := \underbrace{\begin{pmatrix} \mathbf{c}_i(A) & \cdots & \mathbf{c}_i(A) \end{pmatrix}}_{m \text{ identical columns}}.$$

When m, the number of identical rows and columns, is clear from the context, we abbreviate $\mathbf{R}_i^{\times m}(A)$ and $\mathbf{C}_i^{\times m}(A)$ to $\mathbf{R}_i(A)$ and $\mathbf{C}_i(A)$, respectively.

Theorem 2. *Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. For all $i \in [n]$, define*

$$\square_i : \mathcal{PM}(n) \times \mathcal{PM}(m) \to \mathcal{PM}(n+m-1)$$

to be the partial composition maps by

$$A \square_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i(A) & B & O \\ \hline A_{21} & \mathbf{C}_i(A) & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i \end{array}, \tag{4}$$

where A_{11}, A_{21}, A_{22} are the block matrices obtained from A by deleting the ith row and ith column of A. Then the collection $\mathcal{PM} = \bigsqcup_{n\geqslant 1} \mathcal{PM}(n)$ equipped with the partial compositions $\square_i$ forms an operad.

Proof. By construction, for any $i \in [n]$, $A\Box_i B$ is a block lower triangular matrix with diagonal blocks A_{11}, B, A_{22} all poset matrices, which hence have ones on the diagonals. In addition, $A\Box_i B$ preserves transitivity from the transitivity of both A and B. Thus $A\Box_i B$ is a poset matrix in $\mathcal{PM}(n+m-1)$.

To show that the three operad axioms hold on the collection $\mathcal{PM}$, let $A \in \mathcal{PM}(n)$, $B \in \mathcal{PM}(m)$, $C \in \mathcal{PM}(k)$, and choose $i \in [n]$, $j \in [m]$.

(i) *Nested associativity*: Let $M = A\Box_i B \in \mathcal{PM}(n+m-1)$. From (4) we obtain

$$(A\Box_i B)\Box_{i+j-1} C = M\Box_{i+j-1} C = \left(\begin{array}{c|c|c} M_{11} & O & O \\ \hline \mathbf{R}_{i+j-1}(M) & C & O \\ \hline M_{21} & \mathbf{C}_{i+j-1}(M) & M_{22} \end{array}\right), \tag{5}$$

where M_{11}, M_{21} and M_{22} are the block matrices obtained from M by deleting the $(i+j-1)$th row and column. Since $i \leqslant i+j-1 \leqslant i+m-1$, it follows from (4) that

$$\begin{aligned} \mathbf{r}_{i+j-1}(A\Box_i B) &= (a_{i,1}, \ldots, a_{i,i-1} | b_{j,1}, \ldots, b_{j,j-1}) = (\mathbf{r}_i(A) \mid \mathbf{r}_j(B)), \\ \mathbf{c}_{i+j-1}(A\Box_i B) &= (b_{j+1,j}, \ldots, b_{m,j} | a_{i+1,i}, \ldots, a_{ni})^T = (\mathbf{c}_j(B) \mid \mathbf{c}_i(A))^T. \end{aligned}$$

Thus, we obtain

$$\mathbf{R}_{i+j-1}(M) = \left(\mathbf{R}_i^{\times k}(A) \,\middle|\, \mathbf{R}_j^{\times k}(B) \right) \quad \text{and} \quad \mathbf{C}_{i+j-1}(M) = \left(\begin{array}{c} \mathbf{C}_j^{\times k}(B) \\ \hline \mathbf{C}_i^{\times k}(A) \end{array}\right).$$

In addition, it is shown that

$$M_{11} = \left(\begin{array}{c|c} A_{11} & O \\ \hline \mathbf{R}_i^{\times (j-1)}(A) & B_{11} \end{array}\right),$$

$$M_{21} = \left(\begin{array}{c|c} \mathbf{R}_i^{\times (m-j)}(A) & B_{21} \\ \hline A_{21} & \mathbf{C}_i^{\times (j-1)}(A) \end{array}\right),$$

$$M_{22} = \left(\begin{array}{c|c} B_{22} & O \\ \hline \mathbf{C}_i^{\times (m-j)}(A) & A_{22} \end{array}\right),$$

where B_{11}, B_{21}, B_{22} are the block matrices obtained from B by deleting the jth row and column. Substituting these block matrices into (5) yields

$$(A\Box_i B)\Box_{i+j-1} C = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times (m+k-1)}(A) & B\Box_j C & O \\ \hline A_{21} & \mathbf{C}_i^{\times (m+k-1)}(A) & A_{22} \end{array}\right) = A\Box_i (B\Box_j C),$$

which proves the nested associativity.

(ii) *Parallel associativity*: Assume that $i < j$. Then

$$(A\Box_i B)\Box_{j+m-1} C = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times m}(A) & B & O \\ \hline A_{21} & \mathbf{C}_i^{\times m}(A) & A_{22} \end{array}\right) \Box_{j+m-1} C.$$

Since $i+m-1 < j+m-1$, when applying the composition map $\square_{j+m-1}$ to the matrix $A\square_i B$, its first two row blocks remain unchanged, while the bottom row blocks are further transformed through the action of C as follows:

- A_{21} is transformed into $\widehat{A}_{21}$, the matrix obtained from A_{21} by replacing its jth row $(a_{j,1}\ \ldots\ a_{j,i-1})$ with k identical copies of that row;
- $\mathbf{C}_i^{\times m}(A)$ is transformed into $\widehat{\mathbf{C}}_i^{\times m}(A)$, the matrix obtained from $\mathbf{C}_i^{\times m}(A)$ by replacing its jth row $(a_{j,i}\ \ldots\ a_{j,i})$ with k identical copies of that row;
- A_{22} is transformed into $A_{22}\square_j C$.

Thus,

$$(A\square_i B)\square_{j+m-1} C \;=\; \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times m}(A) & B & O \\ \hline \widehat{A}_{21} & \widehat{\mathbf{C}}_i^{\times m}(A) & A_{22}\square_j C \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i+k-1 \end{array}.$$

Let $N = A\square_j C$. Then

$$(A\square_j C)\square_i B = N\square_i B = \left(\begin{array}{c|c|c} N_{11} & O & O \\ \hline \mathbf{R}_i^{\times m}(N) & B & O \\ \hline N_{21} & \mathbf{C}_i^{\times m}(N) & N_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i+k-1 \end{array}$$

where N_{11}, N_{21}, N_{22} are the block matrices obtained from N by deleting the ith row and column. Since $i<j$, it can be easily shown that

$$N_{11} = A_{11}, \quad \mathbf{R}_i^{\times m}(N) = \mathbf{R}_i^{\times m}(A), \quad N_{21} = \widehat{A}_{21}, \quad \mathbf{C}_i^{\times m}(N) = \widehat{\mathbf{C}}_i^{\times m}(A), \quad N_{22} = A_{22}\square_j C.$$

Thus $(A\square_i B)\square_{j+m-1} C = (A\square_j C)\square_i B$, which proves the parallel associativity.
(iii) *Identity*: Let $[1]$ be the 1×1 poset matrix. For all $A \in \mathcal{PM}(n)$ and $i \in [n]$, it is clear that $[1]\square_1 A = A\square_i [1] = A$. Thus, $[1] \in \mathcal{PM}(1)$ acts as the identity element in the collection $\mathcal{PM}$.

From the above results, the partial composition maps defined in (4) endow the combinatorial set of poset matrices $\mathcal{PM}$ with an operad structure. $\square$

The following corollary is an immediate consequence of Theorems 1 and 2. Let $\mathbb{1}_k = (1,\ldots,1)$ and $\mathbf{0}_k = (0,\ldots,0)$ be the row vectors of length k.

Corollary 3. *For $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$, let $A\square_i B$ be the matrix defined in (4).*

(i) *Assume that $A_{21} = J_{n-i,i-1}$.*

  (a) *If $\mathbf{r}_i(A) = \mathbb{1}_{i-1}$ and $\mathbf{c}_i(A) = \mathbb{1}_{n-i}^T$ then $A\square_i B = A \text{▙}_i B$.*

  (b) *If $\mathbf{r}_i(A) = \mathbf{0}_{i-1}$ and $\mathbf{c}_i(A) = \mathbf{0}_{n-i}^T$ then $A\square_i B = A \text{▖}_i B$.*

  (c) *If $\mathbf{r}_i(A) = \mathbb{1}_{i-1}$ and $\mathbf{c}_i(A) = \mathbf{0}_{n-i}^T$ then $A\square_i B = A \text{▌}_i B$.*

(d) *If* $\mathbf{r}_i(A) = \mathbf{0}_{i-1}$ *and* $\mathbf{c}_i(A) = \mathbb{1}_{n-i}^T$ *then* $A\square_i B = A \,\blacksquare\!\!\blacksquare_i\, B$.

(ii) *Assume that* $A_{21} = O_{n-i,i-1}$.

(e) *If* $\mathbf{r}_i(A) = \mathbf{0}_{i-1}$ *and* $\mathbf{c}_i(A) = \mathbf{0}_{n-i}^T$ *then* $A\square_i B = A \boxplus_i B$.

(f) *If* $\mathbf{r}_i(A) = \mathbf{0}_{i-1}$ *and* $\mathbf{c}_i(A) = \mathbb{1}_{n-i}^T$ *then* $A\square_i B = A \boxplus_i B$.

(g) *If* $\mathbf{r}_i(A) = \mathbb{1}_{i-1}$ *and* $\mathbf{c}_i(A) = \mathbf{0}_{n-i}^T$ *then* $A\square_i B = A \boxplus_i B$.

Chain and antichain frequently appear as induced subposets in various posets. In particular, it is notable that $A\square_i B$ may yield the same resulting matrix for different values of i. This suggests a kind of structural redundancy or symmetry in the operation $\square_i$ when applied to specific posets. To illustrate this observation, we consider the chain and antichain posets on $[k]$, and their associated poset matrices denoted by $\mathbb{C}_k$ and $\mathbb{I}_k$, respectively. We now demonstrate, in the following theorem, that different index values in the operation $\square_i$ can result in identical poset matrices.

Theorem 4. *Let $\alpha \subseteq [n]$ be an index set consisting of k consecutive integers. Suppose that $A \in \mathcal{PM}(n)$ is a poset matrix of the block form with $A_{22} = A[\alpha]$:*

$$A = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline A_{21} & A_{22} & O \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right) \tag{6}$$

where row vectors of A_{21} are identical and the column vectors of A_{32} are identical.

(a) *If $A_{22} = \mathbb{C}_k$ then $A\square_i\mathbb{C}_m$ yields the same poset matrix for all $i \in \alpha$;*

(b) *If $A_{22} = \mathbb{I}_k$ then $A\square_i\mathbb{I}_m$ yields the same poset matrix for all $i \in \alpha$.*

Proof. (a) Let $A_{22} = \mathbb{C}_k$. For $i \in \alpha$, assume that all rows of A_{21} are identical to a $(0,1)$-row vector $\mathbf{r}$ of length $i-1$ and all columns of A_{32} are identical to a $(0,1)$-column vector $\mathbf{c}$ of length $n-i$. By the definition of $\square_i$, for all $i \in \alpha$, the poset matrix $A\square_i\mathbb{C}_m \in \mathcal{PM}(n+m-1)$ is of the form:

$$A\square_i\mathbb{C}_m = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \widehat{A}_{21} & \mathbb{C}_{k+m-1} & O \\ \hline A_{31} & \widehat{A}_{32} & A_{33} \end{array}\right),$$

where row vectors of $\widehat{A}_{21}$ and column vectors of $\widehat{A}_{32}$ are respectively identical to $\mathbf{r}$ and $\mathbf{c}$. We observe that A_{11} or A_{33} may be vacuous depending on the choice of the α. Thus, $A\square_i\mathbb{C}_m$ yields the same poset matrix for all $i \in \alpha$, which proves (a).

A similar argument establishes (b). $\square$

For instance, let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{C}_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then for $i = 3, 4$,

$$A\Box_i\mathbb{C}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

It is shown in [6] that every $n \times n$ binary Pascal matrix denoted by $P_n = [b_{ij}]$ where $b_{ij} = \binom{i}{j} \pmod 2$ is a poset matrix. The corresponding poset is called the *Pascal poset* and is denoted by $\mathcal{P}_n$. If $n = 2^k$ then $\mathcal{P}_n$ is isomorphic to the *k-dimensional Boolean lattice* $\mathbb{B}_k$ consisting of all k-subsets ordered by inclusion. As noted in [5], every finite poset is isomorphic to a subposet of a sufficiently large Boolean lattice. It turns out that every finite poset matrix is a submatrix of the infinite binary Pascal matrix. More precisely, every $k \times k$ poset matrix is a submatrix of the $2^k \times 2^k$ binary Pascal matrix. Clearly,

$$P_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Box_2 P_n(1|1),$$

where $P_n(1|1)$ is the $(n-1) \times (n-1)$ matrix obtained by deleting first row and first column of P_n.

We end this section by exploring an additional operad structure on poset matrices. For a poset $P = (X, \preceq)$, an element $a \in P$ is called *minimal* if $x \preceq a$ for some $x \in P$ implies $x = a$. Dually, an element is called *maximal* if $a \preceq x$ implies $x = a$. The following lemma provides a matrix-theoretic interpretation of these notions in terms of the poset matrix.

Lemma 5. *Let P_A be the NL poset on $[n]$ associated with $A \in \mathcal{PM}(n)$.*

(a) *An element $i \in P_A$ is minimal if and only if $i = 1$ or $\mathbf{r}_i(A)$ is the zero vector.*

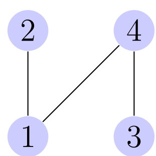
(b) *An element $i \in P_A$ is maximal if and only if $i = n$ or $\mathbf{c}_i(A)$ is the zero vector.*

Proof. (a) Let $A = [a_{ij}] \in \mathcal{PM}(n)$. Since P_A is naturally labelled in $[n]$, an element $i \in P_A$ is minimal if and only if there is no $j \in P_A$ such that $j \prec i$. It follows that $i \in P$ is a minimal element if and only if $i = 1$ or $a_{ij} = 0$ for every $j \in \{1, \ldots, i-1\}$, *i.e.* $\mathbf{r}_i(A)$ is the zero vector.

(b) Similarly, an element $i \in P_A$ is maximal if and only if there is no $j \in P_A$ such that $i \prec j$. It follows that $i \in P_A$ is a maximal element if and only if $i = n$ or $a_{ji} = 0$ for every $j \in \{i+1, \ldots, n\}$, *i.e.* $\mathbf{c}_i(A)$ is the zero vector. □

Example 6. Consider the poset matrix $A \in \mathcal{PM}(4)$ associated to the poset P:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \Leftrightarrow \quad P:$$



Since $\mathbf{r}_3(A) = [0\ 0]$ and $\mathbf{c}_2(A) = [0\ 0]^T$, it follows from Lemma 5 that the minimal elements of P are 1, 3 and the maximal elements of P are 2, 4, which agree with the Hasse diagram of P as shown above.

In the following theorem, we present the additional operad structure on the collection $\mathcal{PM}$ of poset matrices. Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. For each $i \in [n]$, we define the matrix $\mathbf{C}_i(A)^{\min(B)}$ to be the $(n-i) \times m$ matrix whose jth column v_j is given by

$$v_j = \begin{cases} \mathbf{c}_i(A) & \text{if } j \in [m] \text{ is a minimal element of } P_B, \\ \mathbf{0}_{n-i}^T & \text{otherwise.} \end{cases}$$

Thus, $\mathbf{C}_i(A)^{\min(B)}$ encodes copies of the ith sub-column of A at the positions corresponding to the minimal elements of the NL poset P_B associated with the poset matrix B.

Theorem 7. *Let $A \in \mathcal{PM}(n)$, $B \in \mathcal{PM}(m)$. For all $i \in [n]$, define*

$$\circ_i : \mathcal{PM}(n) \times \mathcal{PM}(m) \to \mathcal{PM}(n+m-1)$$

to be the partial composition maps by

$$A \circ_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i(A) & B & O \\ \hline A_{21} & \mathbf{C}_i(A)^{\min(B)} & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i \end{array}, \tag{7}$$

where A_{11}, A_{21}, A_{22} are the block matrices obtained by deleting the ith row and ith column of A. Then the collection $\mathcal{PM} = \bigsqcup_{n \geqslant 1} \mathcal{PM}(n)$ equipped with the partial compositions $\circ_i$ forms an operad.

Proof. Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. The composition $A \circ_i B$ is defined by replacing the i-th row and column of A with B, using the composition rule in (7). Clearly, the resulting matrix is of size $(n+m-1) \times (n+m-1)$, and the transitivity of A and B yields that $A \circ_i B$ is a poset matrix. Thus $A \circ_i B \in \mathcal{PM}(n+m-1)$.

To show that the three operad axioms hold on the collection $\mathcal{PM}$ of poset matrices, we proceed by employing a method analogous to that used in the proof of Theorem 2. Let $A \in \mathcal{PM}(n)$, $B \in \mathcal{PM}(m)$, and $C \in \mathcal{PM}(k)$, and fix indices $i \in [n]$ and $j \in [m]$.

(i) *Nested associativity*: Since $i \leqslant i+j-1 \leqslant i+m-1$, it follows from (7) that $(A \circ_i B) \circ_{i+j-1} C$ is a block matrix of the form

$$\left(\begin{array}{c|c|c|c|c} A_{11} & O & O & O & O \\ \hline \mathbf{R}_i^{\times(j-1)}(A) & B_{11} & O & O & O \\ \hline \mathbf{R}_i^{\times k}(A) & \mathbf{R}_j^{\times k}(B) & C & O & O \\ \hline \mathbf{R}_i^{\times(m-j)}(A) & B_{21} & \mathbf{C}_j^{\times k}(B)^{\min(C)} & B_{22} & O \\ \hline A_{21} & \mathbf{C}_i^{\times(j-1)}(A)^{\min(B)} & \mathbf{C}_i^{\times k}(A)^{\min(C)} & \mathbf{C}_i^{\times(m-j)}(A)^{\min(B)} & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, j-1 \\ \}\, k \\ \}\, m-j \\ \}\, n-i \end{array} \tag{8}$$

where B_{11}, B_{21} and B_{22} are block matrices obtained from B by deleting the jth row and column.

Consider

$$B \mathbin{\square\!\blacksquare}_j C = \left(\begin{array}{c|c|c} B_{11} & O & O \\ \hline \mathbf{r}_j(B) & & \\ \vdots & C & O \\ \mathbf{r}_j(B) & & \\ \hline B_{21} & \mathbf{C}_j^{\times k}(B)^{\min(C)} & B_{22} \end{array}\right) \begin{array}{l} \}\, j-1 \\ \\ \}\, k \\ \\ \}\, m-j \end{array} .$$

An element $\ell \in \{j, j+1, \dots, j+k-1\}$ is a minimal element of $P_{B \mathbin{\square\!\blacksquare}_j C}$ if and only if $\mathbf{r}_j(B) = \mathbf{0}$ and $\mathbf{r}_\ell(C) = \mathbf{0}$ *i.e.*, j and ℓ are minimal elements of P_B and P_C, respectively. Thus, it follows from (8) that

$$\begin{aligned} (A \mathbin{\square\!\blacksquare}_i B) \mathbin{\square\!\blacksquare}_{i+j-1} C &= \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times(m+k-1)}(A) & B \mathbin{\square\!\blacksquare}_j C & O \\ \hline A_{21} & \mathbf{C}_i^{\times(m+k-1)}(A)^{\min(B \mathbin{\square\!\blacksquare}_j C)} & A_{22} \end{array}\right) \\ &= A \mathbin{\square\!\blacksquare}_i (B \mathbin{\square\!\blacksquare}_j C), \end{aligned}$$

which proves the nested associativity.

(ii) *Parallel associativity*: Assume that $i < j$. For the sets $\alpha = \{1, \dots, i-1\}$ and $\beta = \{i+1, \dots, n\}$, we obtain

$$(A \mathbin{\square\!\blacksquare}_i B) \mathbin{\square\!\blacksquare}_{j+m-1} C = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times m}(A) & B & O \\ \hline A_{21} & \mathbf{C}_i^{\times m}(A)^{\min(B)} & A_{22} \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i \end{array} \mathbin{\square\!\blacksquare}_{j+m-1} C. \tag{9}$$

Since $i + m - 1 < j + m - 1$, when applying the composition map $\mathbin{\square\!\blacksquare}_{j+m-1}$ in (9) to the matrix $A \mathbin{\square\!\blacksquare}_i B$, its first two row blocks remain unchanged, while the bottom row blocks are transformed through the action of C as follows:

- A_{21} is transformed into $\widehat{A}_{21}$, the matrix obtained from A_{21} by replacing its jth row $(a_{j,1} \ \dots \ a_{j,i-1})$ with k identical copies of that row;
- $\mathbf{C}_i^{\times m}(A)^{\min(B)}$ is transformed into $\widehat{\mathbf{C}}_i^{\times m}(A)$, the matrix obtained from $\mathbf{C}_i^{\times m}(A)$ by replacing its jth row $(a_{j,i} \ \dots \ a_{j,i})$ with k identical copies of that row;
- A_{22} is transformed into $A_{22} \square_j C$.

Since $i < j$, by a similar method to the proof of parallel associativity in Theorem 2, it is easily shown that

$$\begin{aligned} (A \mathbin{\square\!\blacksquare}_i B) \mathbin{\square\!\blacksquare}_{j+m-1} C &= \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i^{\times m}(A) & B & O \\ \hline \widehat{A}_{21} & \widehat{\mathbf{C}}_i^{\times m}(A) & A_{22} \mathbin{\square\!\blacksquare}_j C \end{array}\right) \begin{array}{l} \}\, i-1 \\ \}\, m \\ \}\, n-i+k-1 \end{array} \\ &= (A \mathbin{\square\!\blacksquare}_j C) \mathbin{\square\!\blacksquare}_i B, \end{aligned}$$

which proves the parallel associativity.

(iii) *Identity*: Let $I_1 = [1]$ be the 1×1 poset matrix. Then for any $A \in \mathcal{PM}(n)$, we have $I_1 \mathbin{\square\!\blacksquare}_1 A = A$. Moreover, I_1 has no minimal elements other than itself so that for any $i \in [n]$, $\mathbf{C}_i^T(A)^{\min(I_1)} = \mathbf{c}_i(A)$. Thus $A \mathbin{\square\!\blacksquare}_i I_1 = A$. $\square$

3 Dual operads of poset matrices

For a poset P, its *dual* P^* is a poset obtained by inverting the order relation of P on the same ground set. A poset P is said to be *self-dual* if P and P^* are isomorphic. The Hasse diagram of P^* is obtained simply by turning the Hasse diagram of P upside down. Clearly, posets appear in dual pairs unless they are self-dual. Thus, the definitions and theorems concerning posets also come in dual pairs; if a theorem holds for all posets, then its dual statement also holds for all posets. From now on, let P be an NL poset on $[n]$, and let A be the corresponding poset matrix. By definition, the dual poset P^* is the NL poset on $[n]$ obtained by relabeling each element $i \in [n]$ with $n-i+1$.

Let $A = [a_{i,j}]$ be an $n\times n$ matrix. The *flip transpose* of A, denoted by A^F, is defined as the matrix obtained by reflecting A over the anti-diagonal (the diagonal from the bottom-left to the top-right corner). Explicitly, if $E = [e_{i,j}]$ denotes the $n\times n$ anti-diagonal matrix with

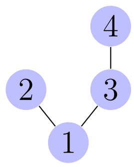
$$e_{i,j} = \begin{cases} 1, & \text{if } i+j = n+1, \\ 0, & \text{otherwise,} \end{cases}$$

then $A^F = EA^TE$, so that A^F is the $n\times n$ matrix whose (i,j)-entry is given by $a_{n-j+1,n-i+1}$ for all $1 \leqslant i,j \leqslant n$. In particular, if $A = A^F$ then the matrix A is symmetric with respect to the anti-diagonal. By the definition of the flip transpose of A we have the following theorem.

Theorem 8. *Let A be the poset matrix associated with an NL poset P on $[n]$. If A^* denotes the poset matrix associated with the dual P^* then $A^* = A^F$.*

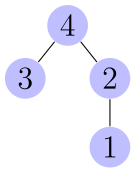
For instance,

P :



$$\Leftrightarrow \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

P^* :



$$\Leftrightarrow \quad A^* = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A^F$$

If $A = [a_{ij}]$ is an $m \times n$ matrix then the flip transpose A^F is defined as the $n \times m$ matrix whose (i,j)-entry is $a_{m-j+1,\,n-i+1}$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. For an index set $\alpha \subset [n]$, let $\alpha^F = \{n-i+1 : i \in \alpha\}$. The following lemma follows from the definition of α^F.

Lemma 9. *Let $A \in \mathcal{PM}(n)$ with the ith sub-row vector $\mathbf{r}_i(A)$ and sub-column vector $\mathbf{c}_i(A)$. Then we have:*

(a) $A^F[\alpha|\beta] = A[\beta^F|\alpha^F]$. *In particular,* $A^F[\alpha] = A[\alpha^F]$.

(b) $\mathbf{r}_i(A)^F = \mathbf{c}_{n-i+1}(A^F)$ *and* $\mathbf{c}_i(A)^F = \mathbf{r}_{n-i+1}(A^F)$.

Similar to the definition of the matrix $\mathbf{C}_i(A)^{\min(B)}$ where $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$, for a fixed $i \in [n]$ let $\mathbf{R}_i(A)^{\max(B)}$ be the $m \times (i-1)$ matrix whose jth row vector u_j is defined by

$$u_j = \begin{cases} \mathbf{r}_i(A) & \text{if } j \in [m] \text{ is a maximal element of } P_B, \\ \mathbf{0}_{i-1} & \text{otherwise.} \end{cases}$$

Accordingly, $\mathbf{R}_i(A)^{\max(B)}$ encodes copies of the ith sub-row vectors of A at the positions corresponding to the maximal elements of the poset P_B. For all $i \in [n]$, define

$$\blacksquare\!\square_i : \mathcal{PM}(n) \times \mathcal{PM}(m) \to \mathcal{PM}(n+m-1)$$

to be the partial composition maps by

$$A \blacksquare\!\square_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i(A)^{\max(B)} & B & O \\ \hline A_{21} & \mathbf{C}_i(A) & A_{22} \end{array}\right). \tag{10}$$

Theorem 10. (Duality) *Let* $A \in \mathcal{PM}(n)$ *and* $B \in \mathcal{PM}(m)$*. Then we have:*

(a) $(A \square_i B)^* = A^* \square_{n-i+1} B^*$.

(b) $(A \square\!\blacksquare_i B)^* = A^* \blacksquare\!\square_{n-i+1} B^*$.

Proof. (a) By Theorem 2, we have

$$A \square_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline U_i & B & O \\ \hline A_{21} & V_i & A_{22} \end{array}\right) \tag{11}$$

where

$$U_i = \mathbf{R}_i(A) = \begin{bmatrix} \mathbf{r}_i(A) \\ \vdots \\ \mathbf{r}_i(A) \end{bmatrix}_{m \times (i-1)} \quad \text{and} \quad V_i = \mathbf{C}_i(A) = \begin{bmatrix} \mathbf{c}_i(A) & \cdots & \mathbf{c}_i(A) \end{bmatrix}_{(n-i) \times m}.$$

By Lemma 9, we have

$$U_i^F = \begin{bmatrix} \mathbf{c}_{n-i+1}(A^F) & \cdots & \mathbf{c}_{n-i+1}(A^F) \end{bmatrix}_{(i-1) \times m} \quad \text{and} \quad V_i^F = \begin{bmatrix} \mathbf{r}_{n-i+1}(A^F) \\ \vdots \\ \mathbf{r}_{n-i+1}(A^F) \end{bmatrix}_{m \times (n-i)}.$$

Thus,

$$(A \square_i B)^* = \left(\begin{array}{c|c|c} A_{22}^F & O & O \\ \hline V_i^F & B^F & O \\ \hline A_{21}^F & U_i^F & A_{11}^F \end{array}\right) = A^F \square_{n-i+1} B^F = A^* \square_{n-i+1} B^*.$$

(b) Let $V_i = \mathbf{C}_i(A)^{\min(B)}$ be given in (11). Since $j \in [m]$ is minimal in the poset P_B if and only if $n - j + 1$ is maximal in its dual $P_B^* = P_{B^F}$, it follows that

$$(A \mathbin{\square\blacksquare}_i B)^* = \left(\begin{array}{c|c|c} A_{22}^F & O & O \\ \hline V_i^F & B^F & O \\ \hline A_{21}^F & U_i^F & A_{11}^F \end{array}\right)$$

where $V_i^F = \mathbf{C}_{n-i+1}(A)^{\max(B)}$. Hence,

$$(A \mathbin{\square\blacksquare}_i B)^* = A^F \mathbin{\blacksquare\square}_{n-i+1} B^F = A^* \mathbin{\blacksquare\square}_{n-i+1} B^*,$$

which proves (b). $\square$

Corollary 11. *Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. For $i \in [n]$, $A \square_i B$ is self dual if and only if the following conditions hold:*

(i) *A and B are self dual;*

(ii) *n is odd and $i = \frac{n+1}{2}$.*

Proof. Assume that $A \square_i B$ is self dual. Then $A \square_i B = (A \square_i B)^F$. Since it is symmetric with respect to the anti-diagonal, it follows from (4) that A and B are self dual. Moreover, using Theorem 10 (a) we have

$$A \square_i B = (A \square_i B)^* = A^* \square_{n-i+1} B^* = A \square_{n-i+1} B,$$

which implies n is odd and $i = \frac{n+1}{2}$. The converse immediately follows from Theorem 10 (a). $\square$

In addition to the operad structure based on minimal elements as shown in Theorem 7, we now present a dual operad structure based on maximal elements of posets.

Theorem 12. (Dual operad structure) *For all $i \in [n]$, let $\mathbin{\blacksquare\square}_i : \mathcal{PM}(n) \times \mathcal{PM}(m) \to \mathcal{PM}(n+m-1)$ be the partial compositions defined by (10). Then the collection $\mathcal{PM} = \bigsqcup_{n \geqslant 1} \mathcal{PM}(n)$ equipped with the partial compositions $\mathbin{\blacksquare\square}_i$ forms an operad.*

Proof. It follows from Theorem 7 and the duality theorem in (b) of Theorem 10. $\square$

Theorem 13. *Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$. For all $i \in [n]$ define*

$$\mathbin{\blacksquare\blacksquare}_i : \mathcal{PM}(n) \times \mathcal{PM}(m) \to \mathcal{PM}(n+m-1)$$

to be the partial composition maps by

$$A \mathbin{\blacksquare\blacksquare}_i B = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \mathbf{R}_i(A)^{\max(B)} & B & O \\ \hline A_{21} & \mathbf{C}_i(A)^{\min(B)} & A_{22} \end{array}\right). \tag{12}$$

Then the partial compositions $\mathbin{\blacksquare\blacksquare}_i$ result in poset matrices for all $i \in [n]$.

Proof. Let $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$ be poset matrices. It follows directly from Theorem 7 and Theorem 12 that for each $i \in [n]$, the operation $\blacksquare_i$ applied to A and B yields

$$A \blacksquare_i B \in \mathcal{PM}(n+m-1).$$

That is, the operation $\blacksquare_i$ preserves the poset matrix property. □

Although the operation $\blacksquare_i$ preserves the structure of poset matrices, it does not in general satisfy the associativity required for an operad. Therefore, this operation does not define an operad structure on the collection $\mathcal{PM}$; see Remark 14 below for a counterexample.

Remark 14. Let A and B be the same poset matrices in Example 18 in Section 4, and let $C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. If $i = 2, j = 3$ then the partial composition $\blacksquare_i$ does not satisfy nested associativity as illustrated below:

$$(A \blacksquare_i B) \blacksquare_{i+j-1} C = \left[\begin{array}{ccc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 \\ 1 & 1 & 0 & \mathbf{0} & 0 & 1 & 1 \end{array}\right]$$

and

$$A \blacksquare_i (B \blacksquare_j C) = \left[\begin{array}{c|cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 \\ 1 & 1 & 0 & \mathbf{1} & 0 & 1 & 1 \end{array}\right].$$

In the study of posets and their matrix representations, localized duality plays a significant role in understanding structural equivalence. We end this section by formalizing a notion of *semi-equiduality* for poset matrices based on duality of a specific principal submatrix corresponding to a disconnected subposet. Let α be a consecutive k-set, *i.e.* it consists of k consecutive elements of $[n]$. For an $n \times n$ matrix A, we denote $A(\alpha)$ the *complementary* submatrix of $A[\alpha]$, *i.e.* it is the $(n-k) \times (n-k)$ submatrix of A obtained by deleting rows and columns indexed by the elements in α. For $A, B \in \mathcal{PM}(n)$, we say that A is a *semi-equidual* of B if there exists a consecutive k-subset $\alpha \subset [n]$ such that $A[\alpha] = (B[\alpha])^F$ *i.e.*, both are dual matrices, whose corresponding NL subposets are disconnected, and $A(\alpha) = B(\alpha)$. The condition that the subposets are disconnected ensures that the duality is localized and non-interfering with the global poset structure.

For example, the following poset matrices A and B are semi-equidual poset matrices with $\alpha = \{2, 3, 4\}$:

$$A = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right).$$

Theorem 15. *Let $\alpha \subseteq [n]$ be a consecutive k-set starting from i. Suppose that $A \in \mathcal{PM}(n)$ is the same block matrix defined in Theorem 4. If $A_{22} = \mathbb{I}_k$ then $A\Box_i\mathbb{C}_m$ is a semi-equidual of $A\Box_{i+k-1}\mathbb{C}_m$.*

Proof. Let $\alpha = \{i, \ldots, i+k-1\} \subseteq [n]$ be a consecutive k-set and let $A_{22} = \mathbb{I}_k$. Similar to the proof of Theorem 4, let $\mathbf{r}$ and $\mathbf{c}$ be the identical row and column vector of A_{21} and A_{32}, respectively. By the definition of $\Box_i$, it is easily shown that

$$A\Box_i\mathbb{C}_m = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \widehat{A}_{21} & \mathbb{C}_m \oplus \mathbb{I}_{k-1} & O \\ \hline A_{31} & \widehat{A}_{32} & A_{33} \end{array}\right) \text{ and } A\Box_{i+k-1}\mathbb{C}_m = \left(\begin{array}{c|c|c} A_{11} & O & O \\ \hline \widehat{A}'_{21} & \mathbb{I}_{k-1} \oplus \mathbb{C}_m & O \\ \hline A_{31} & \widehat{A}'_{32} & A_{33} \end{array}\right),$$

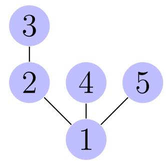
where the row vectors of $\widehat{A}_{21}$ and $\widehat{A}'_{21}$ are all identical to the row vector $\mathbf{r}$, and the column vectors of $\widehat{A}_{32}$ and $\widehat{A}'_{32}$ are all identical to the column vector $\mathbf{c}$. Moreover, it is clear that

$$\mathbb{C}_m \oplus \mathbb{I}_{k-1} = (\mathbb{I}_{k-1} \oplus \mathbb{C}_m)^F,$$

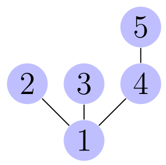
where $\oplus$ denotes the direct sum of two matrices. Moreover, $P_{\mathbb{C}_m\oplus\mathbb{I}_{k-1}}$ and $P_{\mathbb{I}_{k-1}\oplus\mathbb{C}_m}$ are both disconnected. Hence, $A\Box_i\mathbb{C}_m$ is the semi-equidual of $A\Box_{i+k-1}\mathbb{C}_m$. $\Box$

For instance, look at $\alpha = \{2, 3, 4\}$:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \Box_2 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow$$



$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \Box_4 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right) \Leftrightarrow$$



Remark 16. Semi-equidual poset matrices are of particular importance in the recognition of poset isomorphism, a central problem in computational complexity theory. These matrices are characterized by the property that permuting the connected subposet matrices within a principal disconnected subposet matrix preserves the underlying partial order, and the resulting matrix is always isomorphic to the original one. Such invariance is difficult to identify from the graphical structure of posets alone, especially in very large sizes.

The notion of semi-equidual poset matrices therefore provides a meaningful and effective development for studying poset isomorphism from a matrix-theoretic perspective. See [11] for more information on semi-euidual poset matrices.

4 Partial compositions of poset matrices into NL posets

This section translates the partial compositions $\square_i$, $\square\blacksquare_i$, $\blacksquare\square_i$, $\blacksquare\blacksquare_i$ on poset matrices into an explicit construction on naturally labelled posets. We denote by $V(P)$ the set of nodes of an NL poset P.

Theorem 17. *Let $P_A = ([n], \preceq_A)$ and $P_B = ([m], \preceq_B)$ be the NL posets associated with poset matrices $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$, respectively. For a fixed $i \in [n]$ let*

$$\circ_i \in \{\, \square_i,\ \square\blacksquare_i,\ \blacksquare\square_i,\ \blacksquare\blacksquare_i \,\}.$$

Then the poset $P_{A \circ_i B}$ is isomorphic to the poset on the node set $(V(P_A)\backslash\{i\}) \sqcup V(P_B)$, with the partial order $\preceq$ defined as follows:

(i) *If $a, a' \in V(P_A)\backslash\{i\}$, then $a \preceq a' \iff a \preceq_A a'$.*

(ii) *If $b, b' \in V(P_B)$, then $b \preceq b' \iff b \preceq_B b'$.*

(iii) *If $a \in V(P_A)\backslash\{i\}$ and $b \in V(P_B)$, then:*

  (a) *when $\circ_i = \square_i$, $a \preceq b \iff a \preceq_A i$; $b \preceq a \iff i \preceq_A a$;*

  (b) *when $\circ_i = \square\blacksquare_i$, $a \preceq b \iff a \preceq_A i$; $b \preceq a \iff i \preceq_A a$ and $b \in \min(P_B)$;*

  (c) *when $\circ_i = \blacksquare\square_i$, $a \preceq b \iff a \preceq_A i$ and $b \in \max(P_B)$; $b \preceq a \iff i \preceq_A a$;*

  (d) *when $\circ_i = \blacksquare\blacksquare_i$, $a \preceq b \iff a \preceq_A i$ and $b \in \max(P_B)$; $b \preceq a \iff i \preceq_A a$ and $b \in \min(P_B)$.*

Proof. For a fixed $i \in [n]$, let $\circ_i \in \{\square_i,\ \square\blacksquare_i,\ \blacksquare\square_i,\ \blacksquare\blacksquare_i\}$. Clearly, (i) and (ii) follow from the definition of $A \circ_i B$ given in (3). To show (iii), let $a \in V(P_A)\backslash\{i\}$ and $b \in V(P_B)$. Recall that $\mathbf{r}_i(A) = (a_{i,1}, \ldots, a_{i,i-1})$ and $\mathbf{c}_i(A) = (a_{i+1,i}, \ldots, a_{n,i})^T$.

(a) Let $\circ_i = \square_i$. By Theorem 2, U_i is the $m \times (i-1)$ matrix formed by m copies of $\mathbf{r}_i(A)$, and V_i is the $(n-i) \times m$ matrix formed by m copies of $\mathbf{c}_i(A)$. It implies that $a \preceq b$ if and only if $a \preceq_A i$ and $b \preceq a$ if and only if $i \preceq_A a$.

(b) Let $\circ_i = \square\blacksquare_i$. The first assertion follows from (a). By Theorem 7, $V_i = \mathbf{C}_i(A)^{\min(B)}$. Since the jth column vector of V_i is $\mathbf{c}_i(A)$ if and only if j is a minimal element of P_B, it follows that $b \preceq a$ if and only if $i \preceq_A a$ and $b \in \min(P_B)$;

(c) Let $\circ_i = \blacksquare\square_i$. The second assertion follows from (a). By Theorem 12, $U_i = \mathbf{R}_i(A)^{\max(B)}$. Since the jth row vector of U_i is $\mathbf{r}_i(A)$ if and only if j is a maximal element of P_B, it follows that $a \preceq b$ if and only if $a \preceq_A i$ and $b \in \max(P_B)$.

(d) The proof follows from (b) and (c).

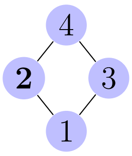
Under the natural relabeling on $[n+m-1]$ of $P_{A\circ_i B}$, the node set $(V(P_A)\setminus\{i\})\sqcup V(P_B)$ is identified with

$$\{1,\dots,i-1\} \sqcup \{i,\dots,i+m-1\} \sqcup \{i+m,\dots,n+m-1\},$$

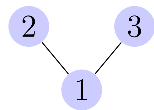
where the set $\{i,\dots,i+m-1\}$ corresponds, in order, to the nodes of P_B, while the remaining elements correspond to the original nodes of P_A with indices shifted accordingly, which completes the proof. $\square$

Example 18. Let $A \in \mathcal{PM}(4)$ and $B \in \mathcal{PM}(3)$ be given as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \mathbf{1} & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Leftrightarrow \quad P_A :$$

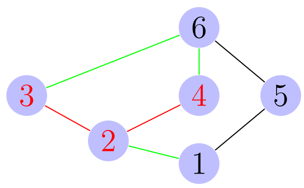


$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Leftrightarrow \quad P_B :$$

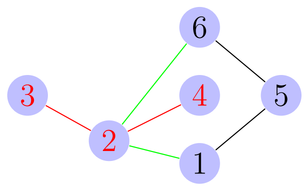


Note that $\mathbf{r}_2(A) = [1]$, $\mathbf{c}_2(A) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in A, the minimal element of P_B is 1 and the maximal elements of P_B are $2,3$. The poset matrices $A \circ_2 B$ with $\circ_2 \in \{\square_2,\ \text{◪}_2,\ \text{◩}_2,\ \blacksquare_2\}$ are constructed according to Theorems 2, 7, 12, and 13. Moreover, by Theorem 17, the associated NL posets $P_{A\circ_2 B}$ can be explicitly obtained from the NL posets P_A and P_B through the corresponding partial compositions as follows:

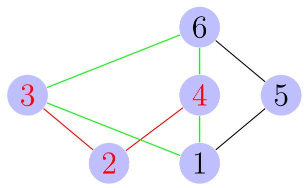
$$A\square_2 B = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \Leftrightarrow$$



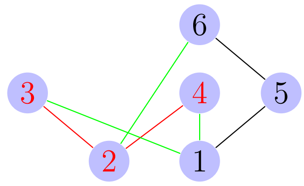
$$A\,\text{◪}_2 B = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \Leftrightarrow$$



$$A\,\text{◩}_2 B = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \Leftrightarrow$$



$$A \boxbslash_2 B = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \Leftrightarrow$$



5 Open Questions

The partial compositions of poset matrices naturally give rise to various enumerative problems. We conclude this paper by raising some open questions concerning the enumeration and structural properties of naturally labeled posets through the operads of poset matrices.

Enumeration via partial compositions. The questions connect the theory of operads of poset matrices with the classical enumeration problem of posets and highlight a possible new operadic approach to Birkhoff-type enumeration problems. Let NL_n denote the set of all NL posets on $[n]$, and let $\circ_i \in \{\square_i, \boxslash_i, \boxbslash_i\}$. Consider a pair $(a,b) \in \mathbb{N} \times \mathbb{N}$ such that $a+b=n+1$.

Problem 19. Given a fixed poset matrix $A \in \mathcal{PM}(a)$ and $i \in [a]$, define the set $\mathcal{F}_n(A, \circ_i)$ by

$$\mathcal{F}_n(A, \circ_i) := \{P_{A \circ_i B} \in \mathsf{NL}_n \mid B \in \mathcal{PM}(b)\} \,/\cong,$$

that is, the set of non-isomorphic NL posets on $[n]$ obtained by composing A with all $B \in \mathcal{PM}(b)$ at position i via the partial composition $\circ_i$. Determine the cardinality $|\mathcal{F}_n(A, \circ_i)|$.

Problem 20. (Birkhoff-type question) Let C be a subset of $\{\square_i, \boxslash_i, \boxbslash_i\}$ and let $\mathcal{F}_n^C$ be the set defined by

$$\mathcal{F}_n^C := \{P_{A \circ_i B} \in \mathsf{NL}_n \mid \circ_i \in C, A \in \mathcal{PM}(a),\ B \in \mathcal{PM}(b),\ i \in [a]\} \,/\cong.$$

Does $\left|\mathcal{F}_n^C\right|$ coincide with the number $g(n)$ of non-isomorphic posets on $[n]$? That is, does the three operadic compositions together of the three operads of poset matrices generate all posets on $[n]$ up to isomorphism? The number $g(n)$ is known up to $n = 16$, see [3] for the complete list. For small values of $g(n)$ (e.g., $g(3) = 5$, $g(4) = 16$), explicit computations demonstrate that $\mathcal{F}_n^C$ coincides with the set of all non-isomorphic posets on $[n]$ with $C = \{\square_i, \boxbslash_i\}$. A detailed list of such constructions for $n = 3, 4$ is provided in Section 6, Appendix.

(a+b)-free posets and their stability. For nonnegative integers a and b, we denote by $\mathbf{a}+\mathbf{b}$ the poset which is the disjoint sum of an a-element chain and a b-element chain. A poset is said to be $(\mathbf{a}+\mathbf{b})$-*free* if it contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. Typical examples include $(\mathbf{2}+\mathbf{2})$-free posets and $(\mathbf{3}+\mathbf{1})$-free posets, which are well-studied in the context of forbidden subposet problems.

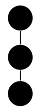
Problem 21. Let P_A and P_B be associated NL posets with $A \in \mathcal{PM}(n)$ and $B \in \mathcal{PM}(m)$ respectivly.

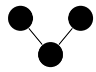
(i) For which pairs $(\mathbf{a}, \mathbf{b})$, if both P_A and P_B are $(\mathbf{a}+\mathbf{b})$-free, is the composed NL poset $P_{A \circ_i B}$ also $(\mathbf{a}+\mathbf{b})$-free?

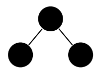
(ii) Fix a pair $(\mathbf{a}, \mathbf{b})$. For which pairs (A, B), if both P_A and P_B are $(\mathbf{a}+\mathbf{b})$-free, does the composed NL poset $P_{A \circ_i B}$ also yield $(\mathbf{a}+\mathbf{b})$-free?

6 Appendix: Operad construction of posets on $[n]$ for $n = 3, 4$

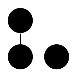
- The 5 nonisomorphic posets with 3 elements:

1. $\begin{pmatrix}1&0\\1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix} \Leftrightarrow$ 

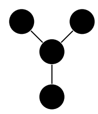
2. $\begin{pmatrix}1&0\\1&1\end{pmatrix} \square_2 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix} \Leftrightarrow$ 

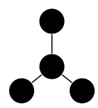
3. $\begin{pmatrix}1&0\\1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix} \Leftrightarrow$ 

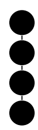
4. $\begin{pmatrix}1&0\\0&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \Leftrightarrow$ 

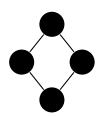
5. $\begin{pmatrix}1&0\\0&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \Leftrightarrow$ 

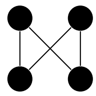
- The 16 nonisomorphic posets with 4 elements:

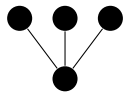
1. $\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix} \square_3 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix} \Leftrightarrow$ 

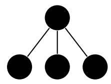
2. $\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix} \square_3 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix} \Leftrightarrow$ 

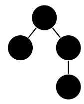
3. $\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix} \Leftrightarrow$ 

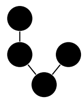
4. $\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix} \square_2 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&1&1&1\end{pmatrix} \Leftrightarrow$ 

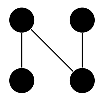
5. $$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1\end{pmatrix} \Leftrightarrow$$ 

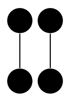
6. $$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix} \square_2 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix} \Leftrightarrow$$ 

7. $$\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\1&1&1&1\end{pmatrix} \Leftrightarrow$$ 

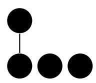
8. $$\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&03\\1&1&1&1\end{pmatrix} \Leftrightarrow$$ 

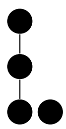
9. $$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix} \square_2 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&0&0&1\end{pmatrix} \Leftrightarrow$$ 

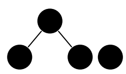
10. $$\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix} \blacksquare\square_2 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&0&0&1\end{pmatrix} \Leftrightarrow$$ 

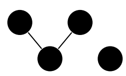
11. $$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \square_3 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix} \Leftrightarrow$$ 

12. $$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \Leftrightarrow$$ 

13. $$\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix} \Leftrightarrow$$ 

14. $$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \square_1 \begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\0&0&0&1\end{pmatrix} \Leftrightarrow$$ 

15. $$\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix} \square_2 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&1&1\end{pmatrix} \Leftrightarrow$$ 

16. $$\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix} \square_3 \begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&0&1\end{pmatrix} \Leftrightarrow$$ 

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