(An algebraic and combinatorial point of view of) Term rewrite systems

INF889K

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Sujets spéciaux en informatique

Hiver 2026

1. General information

1.1. Administrative information

Classes: 15 class meetings, on Mondays, from 13:30 to 16:30.

Evaluation:

- written presentation of an existing research result, 35%;
- oral presentation of an existing research result, 25%;
- exam, 40%.

Each item is scored from 0 to 100.

Passing grade: weighted total score of 50 or higher.

Timetable:

1. **Week 7**: each student must prepare a list of 3 known results on the topic, including a short bibliography.
2. **Week 8**: the professor assigns a result to each student (coming from their list).
3. **Week 13**: each student must write a text presenting the result in LaTeX, 4--16 pages, including the context, proofs, and bibliography.
4. **Week 14**: each student gives an oral presentation of the result, lasting 20 minutes.
5. **Week 15**: exam.

1.2. Content

Objectives:

- Provide an introduction about term rewrite systems;
- Adopt a combinatorial point of view of the topic;
- Apply rewriting techniques for algebraic problems;
- Read, understand, and present research results on the topic.

This course **is not designed to**

- provide an in-depth algorithmic treatment of term rewrite systems;
- cover the categorical viewpoint of the topic.

Content:

1. Abstract rewrite systems (binary relations, first general results);
2. Combinatorics of terms (terms, substitutions);
3. Term series (formal series, products on series, enumeration);
4. Term rewrite systems (matchings, patterns, general definition, main properties);
5. Termination and confluence (reduction orders, polynomial interpretations, critical pairs, completion);
6. Universal algebra and clones (varieties, word problem, Tietze transformations, clones);
7. Programming with term rewriting (applicative systems, currying, combinatory logic).

Bibliography:

1. F. Baader, T. Nipkow, Term Rewriting and All That, Cambridge University Press, 1998.
2. Terese, Term Rewriting Systems, Cambridge University Press, 2003.
3. J. W. Klop, Term Rewriting Systems, Handbook of Logic in Computer Science, Vol. 2, Oxford University Press, 1992.
4. J. R. Hindley, J. P. Seldin, Lambda-Calculus and Combinators, an Introduction, Cambridge University Press, 2008.
5. P. M. Cohn, Universal Algebra, Springer Dordrecht, 1981.

Exercises are classified according to a **difficulty level**:

- ○○○○○ : extremely easy exercise---almost immediate to solve once the question is understood;
- ●○○○○ : very easy exercise---a few minutes to solve and write completely;
- ●●○○○ : easy exercise applying directly the concepts of the lecture---about ten minutes to solve and write completely;
- ●●●○○ : moderate question applying several concepts of the lecture---on the order of an hour to solve and write completely;
- ●●●●○ : difficult question requiring careful consideration---on the order of several hours to solve and write completely;
- ●●●●● : research question requiring a complete exploration---on the order of several days to be considered. In some cases, research questions may still be quite approachable.

1.3. General conventions and notations

Functions are written in **curried form**: given a function

$$f : A_1 \times \cdots \times A_n \to A,$$

we implicitly identify f with its curried form, so that

$$f : A_1 \to \cdots \to A_n \to A$$

where $\to$ is right-associative.

For any $a_1 \in A_1$, the **partial application** $f \cdot a_1$ is the function of type $A_2 \to \cdots \to A_n \to A$ obtained by specializing the first argument of f as a_1. Hence, by iterating this, we write $f \cdot a_1 \cdot \cdots \cdot a_n$ rather than $f(a_1, \ldots, a_n)$.

We will sometimes use **underlining** rather than parentheses to enclose sub-expressions.

Example

Let $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined by $f(x_1, x_2, x_3) := x_1 x_2 + x_1 x_3 + x_2 x_3$.

Under the above identification, the type of f is $\mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ and

$$f \cdot \underline{x + y} \cdot x \cdot \underline{z + t} = (x + y)x + (x + y)(z + t) + x(z + t).$$

The partial application $f \cdot 1$ is the function $g : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ satisfying $g \cdot x_2 \cdot x_3 = x_2 + x_3 + x_2 x_3$.

The *Iverson bracket* is defined as follows. For any statement P,

$$[P] := \begin{cases} 1 & \text{if } P, \\ 0 & \text{otherwise.} \end{cases}$$

Let us define the following **sets of integers**:

- for any $i, j \in \mathbb{Z}$, $[i, j] := \{z \in \mathbb{Z} : i \leqslant z \leqslant j\}$;
- for any $n \in \mathbb{N}$, $[n] := [1, n]$;
- for any $n \in \mathbb{N}$, $[\![n]\!] := \{0\} \cup [n]$.

Some definitions about **words**:

- for any set A, A^* is the set of words on A;
- the **empty word** is denoted by ϵ;
- for any $w \in A^*$, $\ell{\cdot}w$ is the **length** of w;
- for any $w \in A^*$ and $i \in [\ell{\cdot}w]$, $w{\cdot}i$ is the i-th letter of w, where letters are indexed from 1;
- for any $w \in A^*$ and $a \in A$, let $\ell_a{\cdot}w := \#\{i \in [\ell{\cdot}w] : w{\cdot}i = a\}$;
- for any $w, w' \in A^*$, the **concatenation** of w and w' is $w \centerdot w'$.

2. Introduction

2.1. Solving the Coffee Can Problem

Consider the following game, called the *Coffee Can Problem* [D. Gries, The Science of Programming, 1981]:

- let a coffee can containing some white beans ○ and some black beans ●;
- keep an unlimited supply of black beans aside;
- consider the action consisting in randomly picking two beans from the can and
    - if they have the same color, then throw them both and put a black bean into the can;
    - otherwise, throw the black bean and return the white bean into the can;
- repeat this action as long as possible.

Example

Consider the can ○○●●●●○●○.

Pick the 2-nd bean and the 5-th bean. Since they have different colors, the can becomes ○○●●●○●○.

Now, pick the 3-rd and the 4-th bean. Since they have the same color, the can becomes ○○●●○●○.

Some questions about the described process:

- prove that this process always terminates;
- prove that all ways to execute the process lead to the same remaining bean color;
- predict the remaining bean color in terms of the initial can state.

First, we **formalize the problem** by encoding the state of the can as a pair $(i, j) \in \mathbb{N}^2$ where i is the number of $\circ$ and j is the number of $\bullet$ in the can.

Then, we define a **transformation rule** $\Rightarrow$, called *rewrite relation*, which encodes the action:

- when the picked beans are two $\circ$:
$$(i, j) \Rightarrow (i - 2, j + 1);$$
- when the picked beans are two $\bullet$:
$$(i, j) \Rightarrow (i, j - 1);$$
- when the picked beans are of different colors:
$$(i, j) \Rightarrow (i, j - 1).$$

The set of states and of the rewrite relation form a *rewrite system*.

The process terminates because if $(i, j) \Rightarrow (i', j')$, then $i + j > i' + j'$ and, as a state is an element of $\mathbb{N}^2$, it is not possible to perform an infinite sequence of actions from a state.

For this reason, we say that this rewrite system is *terminating*.

To prove that all ways to execute the process lead to the same remaining bean color, we first prove an important property call *local confluence*.

This property holds when, given a state (i, j), if we have two states (i_1, j_1) and (i_2, j_2) such that $(i, j) \Rightarrow (i_1, j_1)$ and $(i, j) \Rightarrow (i_2, j_2)$, there is a state (i', j') reachable from both (i_1, j_1) and (i_2, j_2).

This is the case since we have the commuting square

$$\begin{array}{ccc} (i, j) & \Rightarrow & (i, j-1) \\ \Downarrow & & \Downarrow \\ (i-2, j+1) & \Rightarrow & (i-2, j) \end{array}.$$

This shows that the rewrite system is *locally confluent*.

Since the process is, as shown previously, terminating, by **Newman's Lemma** [M. H. A. Newman, On Theories with a Combinatorial Definition of ''Equivalence'', 1942], the rewrite system is also confluent.

This says exactly that all ways to execute the process lead to the same remaining bean color.

The two possible **final outcomes** of the process are $(1,0)$ and $(0,1)$. They are called *normal forms*.

To predict the final outcome, let us consider the function $\theta : \mathbb{N}^2 \to \mathbb{N}$ such that $\theta \cdot (i,j) := i \mod 2$.

Observe that

$$\theta \cdot (i,j) = \theta \cdot (i, j-1) = i \mod 2$$

and

$$\theta \cdot (i,j) = \theta \cdot (i-2, j+1) = i \mod 2.$$

From this **invariant**, we deduce that

$$(i,j) \Rightarrow^* \begin{cases} (0,1) & \texttt{if } i \mod 2 = 0, \\ (1,0) & \texttt{otherwise.} \end{cases}$$

This shows that the final outcome of the Coffee Can Problem is a white bean if the initial number of white beans in the can is odd, and a black bean otherwise, independently from the choice of each pair of beans at each step of the process.

2.2. A modelization of the Frogs and Toads Puzzle

Let us consider the following **game**, called the *Frogs and Toads Puzzle* [É. Lucas, 1883]:

- let a line of $n + 1 + n$ squares;
- let n frogs $\circ$ on the first n squares;
- let n toads $\bullet$ on the last n squares;
- the middle square is empty $\cdot$;
- the goal consists in exchanging the position of all $\circ$ and $\bullet$ by executing a sequence of moves where
  - a $\circ$ can be moved to its right adjacent square $\cdot$;
  - a $\bullet$ can be moved to its left adjacent square $\cdot$;
  - a $\circ$ can jump over the adjacent $\bullet$ on its right and land on the $\cdot$ on the right;
  - a $\bullet$ can jump over the adjacent $\circ$ on its left and land on the $\cdot$ on the left.

Example

For $n := 3$, we have this sequence of configurations, starting from the initial one:

$$\circ\circ\circ\cdot\bullet\bullet\bullet, \quad \circ\circ\cdot\circ\bullet\bullet\bullet, \quad \circ\circ\bullet\circ\cdot\bullet\bullet, \quad \circ\circ\bullet\circ\bullet\cdot\bullet, \quad \circ\circ\bullet\cdot\bullet\circ\bullet.$$

This puzzle can be formalized in the following way as a **rewrite system on words**.

A **state** is a word on the alphabet $\{\circ, \cdot, \bullet\}$ having n occurrences of $\circ$, 1 occurrence of $\cdot$, and n occurrences of $\bullet$.

Consider the **rewrite rule** $\rightarrow$ on such words, defined by

$$\circ\cdot \rightarrow \cdot\circ,$$

$$\cdot\bullet \rightarrow \bullet\cdot,$$

$$\circ\bullet\cdot \rightarrow \cdot\bullet\circ,$$

$$\cdot\circ\bullet \rightarrow \bullet\circ\cdot.$$

This rewrite rule $\rightarrow$ is extended as a **rewrite relation** $\Rightarrow$ by extending it to the context by

$$uxv \Rightarrow ux'v \text{ if } x \rightarrow x'$$

for any words $u, v \in \{\circ, \cdot, \bullet\}^*$.

The puzzle consists in finding a **rewrite sequence** of the form

$$\circ^n \cdot \bullet^n \Rightarrow \cdots \Rightarrow \bullet^n \cdot \circ^n.$$

Exercise ●●○○○

Let us consider the previous description of the Frog and Toads Puzzle as a rewrite system on words on $n \geqslant 1$ frogs and toads.

1. Describe the **normal forms** of the puzzle, that are, the configurations wherein no move can be performed.
2. Prove that the puzzle is **terminating**, that is, from the initial configuration, by playing a sequence of any moves, a normal form is reached.
3. Prove that the puzzle is **not confluent**, that is, from some configuration, it is possible to play two different moves such that the two resulting configurations lead to no common future configuration.
4. Prove that the required number of moves from the initial configuration to the goal configuration is always $n^2 + 2n$.
5. Provide a description of a sequence of moves from the initial configuration to the goal configuration.

2.3. Computing with natural numbers

Let us represent expressions on natural numbers and a way to compute addition, multiplication, and factorial on these.

Each natural number $n \in \mathbb{N}$ is represented in a *functional way* in the unary numeral system via zero and succ as $\mathsf{succ}^n(\mathsf{zero})$.

Example

The number 4 is represented by

$$\mathsf{succ}^4(\mathsf{zero}) = \mathsf{succ}(\mathsf{succ}(\mathsf{succ}(\mathsf{succ}(\mathsf{zero})))).$$

Each expression on natural numbers involving the operations $+$, $\times$, and $!$ is denoted in a functional way via add, mul, and fact.

Example

The expression $(2 + 3!) \times 5$ is represented functionally by

$$\mathsf{mul}\left(\mathsf{add}\left(\mathsf{succ}^2(\mathsf{zero}), \mathsf{fact}\left(\mathsf{succ}^3(\mathsf{zero})\right)\right), \mathsf{succ}^5(\mathsf{zero})\right).$$

To compute expressions involving addition, let us introduce the rewrite rules

$$\mathsf{add}(n, \mathsf{zero}) \to n,$$

$$\mathsf{add}(n_1, \mathsf{succ}(n_2)) \to \mathsf{succ}(\mathsf{add}(n_1, n_2)).$$

Let $\Rightarrow$ be the rewrite relation obtained by extending $\to$ on the context.

The rewrite relation $\Rightarrow$ is used to locally rewrite an expression into another, while possible, in order to get, from an expression on zero, succ, and add, an expression involving only zero and succ.

Example

We have this rewrite sequence:

$$\mathsf{add}\left(\mathsf{succ}^2(\mathsf{zero}), \mathsf{succ}^3(\mathsf{zero})\right) \Rightarrow \mathsf{succ}\left(\mathsf{add}\left(\mathsf{succ}^2(\mathsf{zero}), \mathsf{succ}^2(\mathsf{zero})\right)\right) \Rightarrow \mathsf{succ}\left(\mathsf{succ}\left(\mathsf{add}\left(\mathsf{succ}^2(\mathsf{zero}), \mathsf{succ}(\mathsf{zero})\right)\right)\right)$$

$$\Rightarrow \mathsf{succ}\left(\mathsf{succ}\left(\mathsf{succ}\left(\mathsf{add}\left(\mathsf{succ}^2(\mathsf{zero}), \mathsf{zero}\right)\right)\right)\right) \Rightarrow \mathsf{succ}\left(\mathsf{succ}\left(\mathsf{succ}\left(\mathsf{succ}^2(\mathsf{zero})\right)\right)\right) = \mathsf{succ}^5(\mathsf{zero}).$$

Exercise ●○○○○

Translate the expression $(1 + 1) + (2 + 1)$ as an expression involving zero, succ, and add, and apply the previous rewrite relation $\Rightarrow$ in order to transform it while possible.

In a similar way, we include the following rules to compute expression involving **multiplications**:

$$\mathsf{mul}(n, \mathsf{zero}) \to \mathsf{zero},$$

$$\mathsf{mul}(n_1, \mathsf{succ}(n_2)) \to \mathsf{add}(\mathsf{mul}(n_1, n_2), n_1),$$

and **factorials**:

$$\mathsf{fact}(\mathsf{zero}) \to \mathsf{succ}(\mathsf{zero}),$$

$$\mathsf{fact}(\mathsf{succ}(n)) \to \mathsf{mul}(\mathsf{succ}(n), \mathsf{fact}(n)).$$

Exercise ●○○○○

Translate the expression $3!$ as a functional expression involving zero, succ, and fact, and apply the six previous rules in order to transform it while possible.

We have defined a **term rewrite system**, where **terms** represent expressions on natural numbers, and the iterated application of the **rewrite relation** allows us to simplify a term into a term which cannot be simplified anymore.

Some questions in this context:

- Some expressions can be rewritten in several different ways. Do all these ways lead to the same result? In other words, is the rewrite system confluent?
- Does this rewrite process always terminate? In other words, is the rewrite system terminating?
- How many steps of rewrites an expression need until the rewriting process ends?
- Does there exist strategies to speed up the process?

2.4. Map operation on lists

Let us represent **expressions on lists** and a way to compute **map** on lists.

Each list is represented in an *applicative way* via nil and cons.

Example

The list $[x_1, x_2, x_3, x_4]$ is represented by

$$\mathsf{cons}\ x_1\ (\mathsf{cons}\ x_2\ (\mathsf{cons}\ x_3\ (\mathsf{cons}\ x_4\ \mathsf{nil}))).$$

Each expression on lists involving the map operation is denoted in an applicative way via map.

Example

The expression specifying the map operation on the list $[1, 2, 3]$ via the factorial function $!$ is represented by

$$\mathsf{map}\ !\ (\mathsf{cons}\ 1\ (\mathsf{cons}\ 2\ (\mathsf{cons}\ 3\ \mathsf{nil}))).$$

To compute such map operation on list, let us introduce the **rewrite rules**

$$\mathsf{map}\ f\ \mathsf{nil} \to \mathsf{nil},$$

$$\mathsf{map}\ f\ (\mathsf{cons}\ x\ l) \to \mathsf{cons}\ (f\ x)\ (\mathsf{map}\ f\ l).$$

Let $\Rightarrow$ be the **rewrite relation** obtained by extending $\to$ on the context.

Example

We have this rewrite sequence:

$$\mathsf{map}\ !\ (\mathsf{cons}\ 1\ (\mathsf{cons}\ 2\ (\mathsf{cons}\ 3\ \mathsf{nil}))) \Rightarrow \mathsf{cons}\ (!\ 1)\ (\mathsf{map}\ !\ (\mathsf{cons}\ 2\ (\mathsf{cons}\ 3\ \mathsf{nil})))$$

$$\Rightarrow \mathsf{cons}\ (!\ 1)\ (\mathsf{cons}\ (!\ 2)\ (\mathsf{map}\ !\ (\mathsf{cons}\ 3\ \mathsf{nil}))) \Rightarrow \mathsf{cons}\ (!\ 1)\ (\mathsf{cons}\ (!\ 2)\ (\mathsf{cons}\ (!\ 3)\ (\mathsf{map}\ !\ \mathsf{nil})))$$

$$\Rightarrow \mathsf{cons}\ (!\ 1)\ (\mathsf{cons}\ (!\ 2)\ (\mathsf{cons}\ (!\ 3)\ (\mathsf{nil}))).$$

Exercise ●●○○○

By using similar methods, propose rules in order to compute left fold on lists by mean of the symbol fold_left.

We have defined an **applicative term rewrite system** to represent lists and their map operation.

In addition to the questions of the previous example, we can ask the following questions:

- What really means ‘‘applicative’’?
- Why such an applicative system is important in this context, as opposed to a functional one?
- Can we represent in a similar way other data structures and their operations, as stacks, heaps, binary search trees, *etc.*?

2.5. Formal derivative of polynomials

The **formal derivative of polynomials** of $\mathbb{K}\langle x \rangle$, where $\mathbb{K}$ is any ring, can be described by the rules

$$\partial k \to 0, \qquad \text{for all } k \in \mathbb{K},$$

$$\partial x \to 1,$$

$$\partial(f_1 + f_2) \to \partial f_1 + \partial f_2,$$

$$\partial(f_1 \times f_2) \to f_1 \times \partial f_2 + \partial f_1 \times f_2.$$

Let $\Rightarrow$ be the **rewrite relation** obtained by extending $\to$ on the context.

Example

We have the following rewrite sequence:

$$\partial\left(x^2 + 2x\right) = \partial(x \times x + 2 \times x) \Rightarrow \partial(x \times x) + \partial(2 \times x) \Rightarrow x \times \partial x + \partial x \times x + \partial(2 \times x)$$

$$\Rightarrow x \times \partial x + \partial x \times x + 2 \times \partial x + \partial 2 \times x \Rightarrow x \times \partial x + \partial x \times x + 2 \times \partial x + 0 \times x$$

$$\Rightarrow x \times \partial x + \partial x \times x + 2 \times \partial x + 0 \times x \Rightarrow x \times 1 + \partial x \times x + 2 \times \partial x + 0 \times x$$

$$\Rightarrow x \times 1 + 1 \times x + 2 \times \partial x + 0 \times x \Rightarrow x \times 1 + 1 \times x + 2 \times 1 + 0 \times x = 2x + 2.$$

We have considered the formal derivative of polynomials as a **rewrite system** allowing us to compute it from any polynomial.

Some observations and questions in this context:

- Here also, the derivative of some polynomials can be executed in different ways. Do all these ways lead to the same result?
- Does this computation always terminate?
- On polynomials, the operations $+$ and $\times$ are associative. This has been assumed implicitly. How to take this property into account rigorously?
- The same question holds for the commutativity for $+$ and $\times$ when $\mathbb{K}$ is a commutative ring.

2.6. Deciding equivalence for groups

A **group** is set together with a binary operation $\star$ which is associative, a neutral element $\mathbb{1}$ w.r.t. $\star$, and an inverse operation $x \mapsto x^{-1}$ w.r.t. $\star$.

A *formal group expression* is an expression combining variables x_i, $i \geqslant 1$, $\star$, $\mathbb{1}$, and $^{-1}$.

Example

$$(x_1 \star x_2) \star \left(x_3 \star \left(x_1^{-1} \star x_3\right)\right)^{-1}$$

is a formal group expression.

A natural question concerns the decision of the equivalence of two formal group expressions.

Two formal group expressions e_1 and e_2 are **equivalent** if in any group, by quantifying universally on the variables appearing in e_1 and e_2, these two expressions compute the same thing.

To prove that two formal group expressions e_1 and e_2 are equivalent, we search for a transformation of e_1 into e_2 by using the group axioms of the *usual presentation of groups*:

$$\mathbb{1} \star x_1 \equiv x_1 \equiv x_1 \star \mathbb{1},$$

$$x_1^{-1} \star x_1 \equiv \mathbb{1} \equiv x_1 \star x_1^{-1},$$

$$(x_1 \star x_2) \star x_3 \equiv x_1 \star (x_2 \star x_3).$$

Example

Let the two formal group expressions $e_1 := (x_1 \star x_2)^{-1}$ and $e_2 := x_2^{-1} \star x_1^{-1}$.
We have

$$e_1 = (x_1 \star x_2)^{-1} \equiv \mathbb{1} \star (x_1 \star x_2)^{-1} \equiv \left(x_2^{-1} \star x_2\right) \star (x_1 \star x_2)^{-1} \equiv x_2^{-1} \star \left(x_2 \star (x_1 \star x_2)^{-1}\right)$$

$$\equiv x_2^{-1} \star \left(\left(x_1^{-1} \star x_1\right) \star \left(x_2 \star (x_1 \star x_2)^{-1}\right)\right) \equiv x_2^{-1} \star \left(x_1^{-1} \star \left(x_1 \star \left(x_2 \star (x_1 \star x_2)^{-1}\right)\right)\right)$$

$$\equiv x_2^{-1} \star \left(x_1^{-1} \star \left((x_1 \star x_2) \star (x_1 \star x_2)^{-1}\right)\right) \equiv x_2^{-1} \star \left(x_1^{-1} \star \mathbb{1}\right) \equiv x_2^{-1} \star x_1^{-1} = e_2,$$

so that e_1 and e_2 are equivalent.

This **does not provide a decision algorithm** since to transform e_1 into e_2, we need to consider the group axioms from left to right or right to left and this could cause an infinite process.

We can deduce from these group axioms the set of ten rules

$$\mathbb{1} \star x_1 \to x_1,$$

$$x_1^{-1} \star x_1 \to \mathbb{1},$$

$$(x_1 \star x_2) \star x_3 \to x_1 \star (x_2 \star x_3),$$

$$x_1^{-1} \star (x_1 \star x_2) \to x_2,$$

$$x_1 \star \mathbb{1} \to x_1,$$

$$\mathbb{1}^{-1} \to \mathbb{1},$$

$$\left((x_1)^{-1}\right)^{-1} \to x_1,$$

$$x_1 \star x_1^{-1} \to \mathbb{1},$$

$$x_1 \star \left(x_1^{-1} \star x_2\right) \to x_2,$$

$$(x_1 \star x_2)^{-1} \to x_2^{-1} \star x_1^{-1}.$$

Let $\Rightarrow$ be the **rewrite relation** obtained by extending $\to$ on the context.

If this rewrite system is **terminating** and **confluent**, then two formal group expressions e_1 and e_2 are equivalent iff their normal forms by $\Rightarrow$ coincide.

Example

Let the two formal group expressions $e_1 := \left(\left(x_1^{-1} \star x_3\right) \star x_3^{-1}\right) \star (x_1 \star x_2)$ and $e_2 := x_1^{-1} \star (x_1 \star x_2 \star \mathbb{1})$.
We have

$$e_1 = \left(\left(x_1^{-1} \star x_3\right) \star x_3^{-1}\right) \star (x_1 \star x_2) \Rightarrow \left(x_1^{-1} \star \left(x_3 \star x_3^{-1}\right)\right) \star (x_1 \star x_2) \Rightarrow \left(x_1^{-1} \star \mathbb{1}\right) \star (x_1 \star x_2) \Rightarrow x_1^{-1} \star (x_1 \star x_2) \Rightarrow x_2$$

and

$$e_2 = x_1^{-1} \star (x_1 \star x_2 \star \mathbb{1}) \Rightarrow x_1^{-1} \star ((x_1 \star x_2) \star \mathbb{1}) \Rightarrow x_1^{-1} \star (x_1 \star (x_2 \star \mathbb{1})) \Rightarrow x_1^{-1} \star (x_1 \star x_2) \Rightarrow x_2,$$

so that, as the same expression x_2 is obtained through the reduction process, e_1 and e_2 are equivalent.

We have described a way to decide the equivalence of expressions in the usual presentation of the group axioms. For this, we have used a **completion** process of these axioms [D. Knuth, P. Bendix, Simple Words Problems in Universal Algebras, 1970].

Some observations and questions in this context:

- How obtain the ten rules from the usual presentation of the group axioms?
- These rules come from [J.-M. Hullot, A catalogue of canonical term rewriting systems, 1980].
- Is this kind of completion possible for any other kinds of algebraic structures (like monoids, bands, lattices, *etc.*)? What are the conditions?

2.7. Alternative axiomatization of commutative groups

Consider **commutative groups**: these are groups such that binary operation $\star$ is commutative.

We can present these algebraic structures by the axioms

$$\mathbb{1} \star x_1 \equiv x_1 \equiv x_1 \star \mathbb{1},$$

$$x_1^{-1} \star x_1 \equiv \mathbb{1} \equiv x_1 \star x_1^{-1},$$

$$(x_1 \star x_2) \star x_3 \equiv x_1 \star (x_2 \star x_3),$$

$$x_1 \star x_2 \equiv x_2 \star x_1.$$

We have in this case

- three generating operations ($\star$, $\mathbb{1}$, and $^{-1}$);
- six axioms.

Is it possible to provide **alternative presentations** for commutative groups?

Let us define the **division** $/$ in commutative groups by

$$x_1/x_2 := x_1 \star x_2^{-1}.$$

We **recover** the usual operations in groups by mean of the division by

- $\mathbb{1} \equiv x_1/x_1$;
- $x_1^{-1} \equiv (x_1/x_1)/x_1$;
- $x_1 \star x_2 \equiv x_1/((x_1/x_1)/x_2)$.

Moreover, we can check that, from the usual group axioms and commutativity of $\star$, we have the **identity**

$$x_1/(x_2/(x_3/(x_1/x_2))) \equiv x_3.$$

The interesting point is that this single identity of which $/$ is subject implies all other axioms of the group in its usual presentation.

Therefore, we can axiomatize commutative groups by mean of a single binary operation and a **single axiom** [A. Tarski, Ein Beitrag zur Axiomatik der Abelschen Gruppen, 1938].

We have provided an alternative description of commutative groups, more minimalistic in some sense compared to the usual one.

Some observations and questions in this context:

- What is the theoretical framework to establish the discussed alternative presentation of commutative groups?
- Can we apply this framework to discover alternative presentations of other algebraic structures? What are the conditions for this to work?
- Can we describe some other algebraic structures only with a single axiom? To give some other interesting examples:
  - this exists for groups [G. Higman, B. H. Neumann, Groups as groupoids with one law, 1952];
  - this exists for lattices [W. McCune, R. Padmanabhan, R. Veroff, Yet another single law for lattices, 2003];
  - this exists for Boolean algebras [W. McCune, R. Veroff, B. Fitelson, K. Harris, A. Feist, L. Wos, Short single axioms for Boolean algebra, 2002];
  - this does not exist for semi-lattices [D. Potts, Axioms for semi-lattices, 1965];
  - this does not exist for distributive lattices [R. McKenzie, Equational Bases for Lattice Theories, 1970].

3. Abstract rewrite systems

3.1. Binary relations

Let X be a set. A *binary relation on* X is a subset $\mathcal{R}$ of X^2.

The property $(x, x') \in \mathcal{R}$ is denoted by $x \,\mathcal{R}\, x'$. The property $(x, x') \notin \mathcal{R}$ is denoted by $x \not\mathcal{R} x'$.

The *identity binary relation on* X is the binary relation

$$\mathcal{I}_X := \{(x, x) : x \in X\}.$$

As binary relations are sets, most of **set operations** can be used on binary relations on a same set X (union, intersection, complement, *etc.*).

The *composition* of two binary relations $\mathcal{R}_1$ and $\mathcal{R}_2$ on X is the binary relation

$$\mathcal{R}_1 \circ \mathcal{R}_2 := \left\{(x, x') \in X^2 : \text{there exists } y \in X \text{ such that } x \,\mathcal{R}_1\, y \text{ and } y \,\mathcal{R}_2\, x'\right\}.$$

This operation is **associative** and admits $\mathcal{I}_X$ as the **neutral element**.

Our convention for composition of binary relations is **left-to-right** (opposite to the usual convention for function composition).

Let $\mathcal{R}$ be a binary relation on X.

The *inverse* of $\mathcal{R}$ is the binary relation

$$\mathcal{R}^{-1} := \left\{(x, x') \in X^2 : x' \,\mathcal{R}\, x\right\}.$$

For any $k \in \mathbb{Z}$, the *k-th composition* of $\mathcal{R}$ is the binary relation

$$\mathcal{R}^k := \begin{cases} \mathcal{R}^{k-1} \circ \mathcal{R} & \text{if } k \geqslant 1, \\ \mathcal{I}_X & \text{if } k = 0, \\ \mathcal{R}^{-1} \circ \mathcal{R}^{k+1} & \text{otherwise } (k \leqslant -1). \end{cases}$$

Let us consider the following **closures**:

- the *reflexive closure* of $\mathcal{R}$, defined as $\mathcal{R}^0 \cup \mathcal{R}$;
- the *symmetric closure* of $\mathcal{R}$, defined as $\mathcal{R} \cup \mathcal{R}^{-1}$;
- the *transitive closure* of $\mathcal{R}$, defined as $\mathcal{R}^+ := \bigcup_{n \geqslant 1} \mathcal{R}^n$;
- the *reflexive and transitive closure* of $\mathcal{R}$, defined as $\mathcal{R}^* := \mathcal{R}^0 \cup \mathcal{R}^+$;
- the *reflexive, symmetric, and transitive closure* of $\mathcal{R}$, defined as $\mathcal{R}^\bullet := \left(\mathcal{R} \cup \mathcal{R}^{-1}\right)^*$.

3.2. Abstract rewrite systems

Definition

An *abstract rewrite system* (*ARS*) is a pair $(X, \Rightarrow)$ such that X is a set, called the *underlying set*, and $\Rightarrow$ is a binary relation on X, called the *rewrite relation*.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

Let us introduce some notations:

- let $\Leftarrow$ be the **inverse** $\Rightarrow^{-1}$ of $\Rightarrow$.
  The ARS $(X, \Leftarrow)$ is the *dual* of $\mathcal{A}$;
- let $\Leftrightarrow$ be the **symmetric closure** of $\Rightarrow$;
- let $\equiv$ be the **reflexive, symmetric, and transitive closure** $\Rightarrow^{\bullet}$ of $\Rightarrow$.
  The **equivalence relation** $\equiv$ on X is the *convertibility relation* of $\mathcal{A}$.
  The $\equiv$-**equivalence class** of $x \in X$ is denoted by $[x]_{\equiv}$.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

- The *set of successors* of $x \in X$ in $\mathcal{A}$ is the set $x^{\Rightarrow} := \{x' \in X : x \Rightarrow x'\}$.
  For any $x' \in x^{\Rightarrow}$, x *one-step rewrites* to x'.
  The function $x \mapsto x^{\Rightarrow} : X \to \mathcal{P} \cdot X$ is the *successor function* of $\mathcal{A}$.
- The *set of predecessors* of $x \in X$ in $\mathcal{A}$ is the set $x^{\Leftarrow}$ of successors of x in the dual of $\mathcal{A}$.
  The *predecessor function* of $\mathcal{A}$ is the successor function of the dual of $\mathcal{A}$.
- The *future* of $x \in X$ in $\mathcal{A}$ is the set $x^{\Rightarrow^*}$.
  For any $x' \in x^{\Rightarrow^*}$, x *rewrites* to x'.
  The *future function* of $\mathcal{A}$ is the successor function of $(X, \Rightarrow^*)$.
- The *past* of $x \in X$ in $\mathcal{A}$ is the set $x^{\Leftarrow^*}$ of the future of x in the dual of $\mathcal{A}$.
  The *past function* of $\mathcal{A}$ is the future function of the dual of $\mathcal{A}$.

The *rewrite graph* of $\mathcal{A}$ is the **directed graph** having X as set of vertices and $\Rightarrow$ as set of arcs.

An ARS can be **specified equivalently** through its rewrite relation, through its successor function, or through its predecessor function.

Example

Let the ARS $\mathsf{Succ} := (\mathbb{N}, \Rightarrow)$ such that $n \Rightarrow n+1$ for any $n \in \mathbb{N}$. In Succ, we have for any $n \in \mathbb{N}$,

- $n^{\Rightarrow} = \{n+1\}$;
- $n^{\Rightarrow^*} = \{m \in \mathbb{N} : m \geqslant n\}$;
- $n^{\Leftarrow} = \{n-1\}$ if $n \geqslant 1$ and $0^{\Leftarrow} = \emptyset$;
- $n^{\Leftarrow^*} = \{m \in \mathbb{N} : m \leqslant n\}$.

Example

Let the ARS Pred defined as the dual of Succ. In Pred, we have for any $n \in \mathbb{N}$,

- $n^{\Rightarrow} = \{n-1\}$ if $n \geqslant 1$ and $0^{\Rightarrow} = \emptyset$;
- $n^{\Rightarrow^*} = \{m \in \mathbb{N} : m \leqslant n\}$;
- $n^{\Leftarrow} = \{n+1\}$;
- $n^{\Leftarrow^*} = \{m \in \mathbb{N} : m \geqslant n\}$.

Example

Let the ARS $\mathsf{PredSucc} := (\mathbb{N}, \Rightarrow)$ such that $n \Rightarrow n + 1$ for any $n \in \mathbb{N}$ and $n \Rightarrow n - 1$ for any $n \in \mathbb{N} \setminus \{0\}$. In $\mathsf{PredSucc}$, we have for any $n \in \mathbb{N}$,

- $n^{\Rightarrow} = \{n - 1, n + 1\}$ if $n \geqslant 1$ and $0^{\Rightarrow} = \{1\}$;
- $n^{\Rightarrow^*} = \mathbb{N}$;
- $n^{\Leftarrow} = n^{\Rightarrow}$;
- $n^{\Leftarrow^*} = \mathbb{N}$.

Example

Let the ARS $\mathsf{Factors} := (\mathbb{N} \setminus \{0\}, \Rightarrow)$ such that $n \Rightarrow m$ if $m \in \mathbb{N}$ is a (proper or not) factor of $n \in \mathbb{N}$. In $\mathsf{Factors}$, we have for any $n \in \mathbb{N}$,

- $n^{\Rightarrow} = \{m \in \mathbb{N} \setminus \{0\} : m \texttt{ divides } n\}$;
- $n^{\Rightarrow^*} = n^{\Rightarrow}$;
- $n^{\Leftarrow} = \{nm : m \in \mathbb{N} \setminus \{0\}\}$;
- $n^{\Leftarrow^*} = n^{\Leftarrow}$.

Example

Let for any $k \geqslant 1$ the ARS $\mathsf{Cycle}_k := ([\![k-1]\!], \Rightarrow)$ such that $n \Rightarrow n+1 \mod k$ for any $n \in [\![k-1]\!]$. In Cycle_k, we have for any $n \in [\![k-1]\!]$,

- $n^{\Rightarrow} = \{n+1 \mod k\}$;
- $n^{\Rightarrow^*} = [\![k-1]\!]$;
- $n^{\Leftarrow} = \{n-1 \mod k\}$;
- $n^{\Leftarrow^*} = [\![k-1]\!]$.

Example

Let the ARS $\mathsf{Grid} := \left(\mathbb{Z}^2, \Rightarrow\right)$ such that $(i,j) \Rightarrow (i+1, j)$ and $(i,j) \Rightarrow (i, j+1)$ for any $i, j \in \mathbb{Z}$. In Grid, we have for any $(i,j) \in \mathbb{Z}^2$,

- $(i,j)^{\Rightarrow} = \{(i+1, j), (i, j+1)\}$;
- $(i,j)^{\Rightarrow^*} = \left\{(i', j') \in \mathbb{Z}^2 : i' \geqslant i \text{ and } j' \geqslant j\right\}$;
- $(i,j)^{\Leftarrow} = \{(i-1, j), (i, j-1)\}$;
- $(i,j)^{\Leftarrow^*} = \left\{(i', j') \in \mathbb{Z}^2 : i' \leqslant i \text{ and } j' \leqslant j\right\}$.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

In $\mathcal{A}$, $x \in X$ is

- *finitely branching* if $x^{\Rightarrow}$ is **finite**;
- *globally finite* if $x^{\Rightarrow^*}$ is **finite**;
- *acyclic* if $x \notin x^{\Rightarrow^+}$.

When all elements of X are finitely branching (resp. globally finite, acyclic), $\mathcal{A}$ is *finitely branching* (resp. *globally finite*, *acyclic*).

Examples

- The ARS Succ is finitely branching and not globally finite. Since for any $n \in \mathbb{N}$, $n^{\Rightarrow^+} = \{m \in \mathbb{N} : m > n\}$, Succ is acyclic.
- The ARS Pred is finitely branching and globally finite. Since for any $n \in \mathbb{N}$, $n^{\Rightarrow^+} = \{m \in \mathbb{N} : m < n\}$, Pred is acyclic.
- For any $k \geqslant 1$, the ARS Cycle_k is finitely branching and globally finite. Since, for any $n \in [\![k-1]\!]$, $n^{\Rightarrow^+} = [\![k-1]\!]$, Cycle_k is not acyclic.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

Let I be a **nonempty initial interval** of $\mathbb{N}$ (possibly infinite).

A *rewrite sequence* in $\mathcal{A}$ is a sequence $u = (u_i)_{i \in I}$ on X such that for any $i \in I$, if $i + 1 \in I$, then $u_i \Rightarrow u_{i+1}$.

If $u = (u_i)_{i \in I}$ is a rewrite sequence in $\mathcal{A}$:

- u *starts from* u_0;
- when I is finite, u *ends at* u_ℓ where ℓ is the greatest element of I;
- when I is finite, the *length* $\ell \cdot u$ of u is $\# I - 1$, that is, the greatest element of I.

Example

The sequence

$$(0, -1)\,(0, 0)\,(0, 1)\,(1, 1)\,(2, 1)\,(3, 1)$$

is a rewrite sequence in Grid starting from $(0, -1)$, ending at $(3, 1)$, and of length 5.

Moreover,

$$(1, 1)\,(1, 2)\,(1, 3)\,(1, 4)\ldots$$

is an infinite rewrite sequence in Grid starting from $(1, 1)$.

Let $\mathcal{A} := (X, \Rightarrow)$ and $\mathcal{A}' := (X', \Rightarrow')$ be two ARSs.

1. If $X' \subseteq X$ and $\Rightarrow' \subseteq \Rightarrow \cap X'^2$, then $\mathcal{A}'$ is a *sub-ARS* of $\mathcal{A}$.
2. If $\mathcal{A}'$ is a sub-ARS of $\mathcal{A}$ and $\Rightarrow' = \Rightarrow \cap X'^2$, then $\mathcal{A}'$ is an *induced sub-ARS* of $\mathcal{A}$.
3. If $\mathcal{A}'$ is an induced sub-ARS of $\mathcal{A}$ and for any $x \in X$ and $x' \in X'$, $x' \Rightarrow x$ implies $x \in X'$, then $\mathcal{A}'$ is a *closed sub-ARS* of $\mathcal{A}$.

Examples

1. The ARS $\left(\mathbb{N}^2, \Rightarrow\right)$ such that $(i, j) \Rightarrow (i + 1, j)$ for any $i, j \in \mathbb{N}$ is a sub-ARS of Grid.
2. The ARS $(\{(0,0), (1,0), (0,1), (1,1)\}, \Rightarrow)$ such that $(0,0) \Rightarrow (1,0)$, $(0,0) \Rightarrow (0,1)$, $(1,0) \Rightarrow (1,1)$, and $(0,1) \Rightarrow (1,1)$ is an induced sub-ARS of Grid.
3. The ARS $(\{(i,j) : i, j \geqslant 1\}, \Rightarrow)$ such that $(i,j) \Rightarrow (i+1, j)$ and $(i,j) \Rightarrow (i, j+1)$ for any $i, j \geqslant 1$ is a closed sub-ARS of Grid.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

Given $Y \subseteq X$, the *closed sub-ARS of $\mathcal{A}$ generated by Y* is the smallest closed sub-ARS w.r.t. inclusion of $\mathcal{A}$ such that underlying set contains Y.

Proposition [Closed sub-ARS generated by a set]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS and Y be a subset of X. The closed sub-ARS of $\mathcal{A}$ generated by Y is the ARS $(Y', \Rightarrow')$ where $Y' := \bigcup_{y \in Y} y^{\Rightarrow^*}$ and $\Rightarrow' := \Rightarrow \cap Y'^2$.

Exercise ●●○○○

Prove the previous proposition.

Example

The closed sub-ARS of Grid generated by $\{(-3, 4), (2, -1)\}$ is the ARS $(X, \Rightarrow)$ such that $X = \left\{(i, j) \in \mathbb{Z}^2 : (i \geqslant -3 \text{ and } j \geqslant 4) \text{ or } (i \geqslant 2 \text{ and } j \geqslant -1)\right\}$, and $\Rightarrow$ is the restriction of the rewrite relation of Grid to X^2.

Exercise ●○○○○

Let S be a set and $\mathsf{Sets}_S := (\mathcal{P} \cdot S, \Rightarrow)$ be the ARS such that $Z \Rightarrow Z'$ if $Z' = Z \sqcup \{s\}$ for an $s \in S \setminus Z$.

For instance, in $\mathsf{Sets}_{\mathbb{N}}$,

$$\{1,4\} \Rightarrow \{1,4,5\} \Rightarrow \{1,3,4,5\} \Rightarrow \{1,3,4,5,9\}.$$

1. Rephrase the definition of Sets_S by using a successor function.
2. Describe a necessary and sufficient condition for the property of Sets_S to be finitely branching.
3. Describe the dual of Sets_S by giving its successor function.
4. Describe the future function of Sets_S.
5. Prove that Sets_S is acyclic.
6. Define a sub-ARS of Sets_S which is not an induced sub-ARS of Sets_S.
7. Define an induced sub-ARS of Sets_S which is not a closed sub-ARS of Sets_S.
8. Define a closed sub-ARS of Sets_S which is not Sets_S itself.

Exercise ●●●○○

Let, for any $n \in \mathbb{N}$, $\mathfrak{S}_n$ be the set of permutations of size n. For instance,

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Let $\mathsf{Permutations}_n := (\mathfrak{S}_n, \Rightarrow)$ be the ARS such that $\sigma \Rightarrow \sigma'$ if $\sigma' = \sigma \circ s_i$ where $i \in [n-1]$, s_i is an elementary transposition of $\mathfrak{S}_n$, and $\sigma \cdot i < \sigma \cdot \lfloor i+1 \rfloor$.

For instance, in $\mathsf{Permutations}_6$, we have $452361 \Rightarrow 452631$ since $452631 = 452361 \circ s_4$ and $452361 \cdot 4 = 3 < 6 = 452361 \cdot 5$. Besides, in $\mathsf{Permutations}_7$, we have

$$3714652 \Rightarrow 7314652 \Rightarrow 7316452 \Rightarrow 7316542.$$

1. Rephrase the definition of $\mathsf{Permutations}_n$ by using a successor function.
2. Prove that all rewrite sequences in $\mathsf{Permutations}_n$ are finite.
3. Describe the future function of $\mathsf{Permutations}_n$. In other terms, provide a necessary and sufficient combinatorial criterion on $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_n$ for the property $\sigma \Rightarrow^* \sigma'$, without having to exhibit a rewrite sequence starting from σ and ending at σ'.

Abstract rewrite systems

3.3. Termination

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

An $x \in X$ is a *normal form* of $\mathcal{A}$ if $x^{\Rightarrow} = \emptyset$.

A rewrite sequence u in $\mathcal{A}$ is *normalizing* if u is **finite** and **ends at a normal form** of $\mathcal{A}$.

In $\mathcal{A}$, $x \in X$ is

- *non-normalizing* if $x^{\Rightarrow^*}$ contains no normal form of $\mathcal{A}$;
- *normalizing* if there exists a normalizing rewrite sequence in $\mathcal{A}$ which starts from x;
- *terminating* if all rewrite sequences starting from x in $\mathcal{A}$ are finite.

Note that in $\mathcal{A}$,

- the set of **normal forms** is a subset of the set of **terminating** elements;
- the set of **terminating** elements is a subset of the set of **normalizing** elements;
- the set of **non-normalizing** elements is disjoint from the set of **normalizing** elements.

When all elements of X are normalizing (resp. terminating), $\mathcal{A}$ is *normalizing* (resp. *terminating*).

Example

The only normal form of Pred is 0. Since for any $n \in \mathbb{N}$, any rewrite sequence starting from n is of the form $n, n-1, \dots, m$ with $0 \leqslant m \leqslant n$, n is terminating. Therefore, Pred is terminating.

Example

Let the ARS $\mathsf{EvenOdd} := (\mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}, \Rightarrow)$ such that, for any $n \in \mathbb{N}$, $n \Rightarrow n+1$, $n \Rightarrow \mathrm{e}$ if n is even, and $n \Rightarrow \mathrm{o}$ if n is odd. The only normal forms of $\mathsf{EvenOdd}$ are e and o. Since for any $n \in \mathbb{N}$, either $n\mathrm{e}$ or $n\mathrm{o}$ is a normalizing rewrite sequence in $\mathsf{EvenOdd}$ starting from n, n is normalizing. Moreover, since $n, n + 1, \dots$ is an infinite rewrite sequence in $\mathsf{EvenOdd}$, n is not terminating. Therefore, $\mathsf{EvenOdd}$ is normalizing and not terminating.

Example

Let $\mathcal{A} := (\{a, b, c\}, \Rightarrow)$ be the ARS such that $a \Rightarrow b$, $a \Rightarrow c$, and $b \Rightarrow b$. Since $c^{\Rightarrow} = \emptyset$, c is a normal form of $\mathcal{A}$. Besides, since ac is a normalizing rewrite sequence in $\mathcal{A}$, a is normalizing. Moreover, since ab^{ω} is an infinite rewrite sequence in $\mathcal{A}$ (b^{ω} denotes the infinite sequence made of b), a is not terminating. Observe that b is not normalizing since the only rewrite sequences starting from b in $\mathcal{A}$ are of the form b^n, $n \in \mathbb{N}$, or b^{ω}. Therefore, $\mathcal{A}$ is not normalizing.

3.4. Confluence

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

For any $x, x' \in X$, let

$$x \Downarrow x' := x^{\Rightarrow^*} \cap x'^{\Rightarrow^*}.$$

When $x \Downarrow x' \neq \emptyset$, x and x' are *joinable* in $\mathcal{A}$.

Observe that x and x' are joinable in $\mathcal{A}$ iff x and x' have a common element in their futures.

Let $\bar{\Downarrow} := \Rightarrow^* \circ \Leftarrow^*$ be the *joinability relation* of $\mathcal{A}$.

Proposition [Joinability relation]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. For any $x, x' \in X$, x and x' are joinable in $\mathcal{A}$ iff $x \bar{\Downarrow} x'$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

For any $x, x' \in X$, let

$$x \Uparrow x' := x^{\Leftarrow^*} \cap x'^{\Leftarrow^*}.$$

When $x \Uparrow x' \neq \emptyset$, x and x' are *meetable* in $\mathcal{A}$.

Observe that x and x' are meetable in $\mathcal{A}$ iff x and x' have **a common element in their pasts**.

Let $\bar{\Uparrow} := \Leftarrow^* \circ \Rightarrow^*$ be the *meetability relation* of $\mathcal{A}$.

Proposition [Meetability relation]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. For any $x, x' \in X$, x and x' are meetable in $\mathcal{A}$ iff $x \mathrel{\bar{\Uparrow}} x'$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *confluent* if $\bar{\Uparrow} \subseteq \bar{\Downarrow}$.

Proposition [Confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow^* y$ and $x \Rightarrow^* y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ has the *Church-Rosser property* if $\equiv \subseteq \overline{\Downarrow}$.

Proposition [Church-Rosser property]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ has the Church-Rosser property;
2. for any $y, y' \in X$, if $y \equiv y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *semi-confluent* if $\Leftarrow \circ \Rightarrow^* \subseteq \bar{\Downarrow}$.

Proposition [Semi-confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is semi-confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow y$ and $x \Rightarrow^* y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Theorem [Semi-confluence, confluence, and Church-Rosser property]

Let $\mathcal{A}$ be an ARS. The following three properties are equivalent:

1. $\mathcal{A}$ is semi-confluent;
2. $\mathcal{A}$ is confluent;
3. $\mathcal{A}$ has the Church-Rosser property.

Proof. Assume that $\mathcal{A} = (X, \Rightarrow)$. By using Propositions [Church-Rosser property], [Confluence], and [Semi-confluence], we have that 3. implies 2., and that 2. implies 1..

Conversely, assume that 1. holds. Let $y, y' \in X$ such that $y \equiv y'$. There exists $k \geqslant 0$ and $y_0, y_1, \dots, y_k \in X$ such that $y = y_0 \Leftrightarrow y_1 \Leftrightarrow \cdots \Leftrightarrow y_k = y'$. We prove $y \mathrel{\overline{\Downarrow}} y'$ by induction on k. If $k = 0$, $y = y'$ and the property holds immediately. Otherwise, $k \geqslant 1$ and, by induction hypothesis, $y \mathrel{\overline{\Downarrow}} y_{k-1}$. Thus, there exists $z \in X$ such that $y \Rightarrow^* z$ and $y_{k-1} \Rightarrow^* z$. We have now two cases:

- If $y_{k-1} \Rightarrow y'$, then, as we have both $y_{k-1} \Rightarrow^* z$ and $y_{k-1} \Rightarrow y'$, by Proposition [Semi-confluence], $z \mathrel{\overline{\Downarrow}} y'$. Since $y \Rightarrow^* z$, this implies $y \mathrel{\overline{\Downarrow}} y'$.
- Otherwise, $y' \Rightarrow y_{k-1}$. We have $y' \Rightarrow^* z$ and $y \Rightarrow^* z$ so that $y \mathrel{\overline{\Downarrow}} y'$.

This shows that $y \equiv y'$ implies $y \mathrel{\overline{\Downarrow}} y'$, proving that $\mathcal{A}$ has the Church-Rosser property. Therefore, 1. implies 3..

Proposition [Finite Church-Rosser property]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS having the Church-Rosser property. For any $y_1, \ldots, y_n \in X$, $n \in \mathbb{N}$, if $y_i \equiv y_j$ for all $i, j \in [n]$, then there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n]$.

Proof. We proceed by induction on n. The property holds vacuously when $n = 0$. When $n = 1$, $z = y_1$ satisfies the property. When $n = 2$, this is exactly the property stated by Proposition [Church-Rosser property]. Assume now that $n \geqslant 3$ and $y_i \equiv y_j$ for all $i, j \in [n]$. By induction hypothesis, there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n-1]$. Now, since $z \equiv y_n$, by Proposition [Church-Rosser property], there exists $z' \in X$ such that $z \Rightarrow^* z'$ and $y_n \Rightarrow^* z'$. As $y_i \Rightarrow^* z'$ for any $i \in [n]$, the desired property is established.

Note that this result, by using Proposition [Confluence] and Theorem [Semi-confluence, confluence, and Church-Rosser property] implies an analogous property for confluence, the **finite confluence property**: If $\mathcal{A} := (X, \Rightarrow)$ is a confluent ARS, then for any $x, y_1, \ldots, y_n \in X$, $n \in \mathbb{N}$, the property $x \Rightarrow^* y_i$ for all $i \in [n]$ implies that there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n]$.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *locally confluent* if $\Leftarrow \circ \Rightarrow \subseteq \overline{\Downarrow}$.

Proposition [Local confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is locally confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow y$ and $x \Rightarrow y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

In contrast with Theorem [Semi-confluence, confluence, and Church-Rosser property], a locally confluent ARS is not necessarily confluent.

Example

Let us consider the ARS EvenOdd defined previously.

For any $y, y' \in \mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}$, $y \, (\Leftarrow \circ \Rightarrow) \, y'$ iff there exists $x \in \mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}$ such that $y \Leftarrow x$ and $x \Rightarrow y'$.

We have one of the following possibilities:

1. $y = n + 1$, $x = n$, $y' = \mathrm{e}$, $n \in \mathbb{N}$ even;
2. $y = \mathrm{e}$, $x = n$, $y' = n + 1$, $n \in \mathbb{N}$ even;
3. $y = \mathrm{e}$, $x = n$, $y' = \mathrm{e}$, $n \in \mathbb{N}$ even;

1. $y = n + 1$, $x = n$, $y' = \mathrm{o}$, $n \in \mathbb{N}$ odd;
2. $y = \mathrm{o}$, $x = n$, $y' = n + 1$, $n \in \mathbb{N}$ odd;
3. $y = \mathrm{o}$, $x = n$, $y' = \mathrm{o}$, $n \in \mathbb{N}$ odd.

In Cases 1., 2., and 3. of the left, we have $\mathrm{e} \in y \Downarrow y'$.

In Cases 1., 2., and 3. of the right, we have $\mathrm{o} \in y \Downarrow y'$.

In each case, $y \bar{\Downarrow} y'$ holds so that EvenOdd is locally confluent.

Since $\mathrm{o} \Leftarrow^* 0 \Rightarrow^* \mathrm{e}$, and e and o are not joinable, EvenOdd is not confluent.

Abstract rewrite systems

3.5. General properties

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The set of **normal forms** in the **future** of $x \in X$ in $\mathcal{A}$ is denoted by $x^{\Rightarrow}$.

The *normal form function* of $\mathcal{A}$ is the function $x \mapsto x^{\Rightarrow}$.

Proposition [Termination and normal forms]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is terminating, then for any $x \in X$, $x^{\Rightarrow} \neq \emptyset$.

Exercise ●○○○○

Prove the previous proposition.

Proposition [Confluence and normal forms]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is confluent, then for any $x \in X$, $x^{\Rightarrow}$ is empty or is a singleton.

Exercise ●●○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ and $\mathcal{A}' := (X, \Rightarrow')$ be two ARSs.

The ARSs $\mathcal{A}$ and $\mathcal{A}'$ are

- *future equivalent* if the **future functions** of $\mathcal{A}$ and $\mathcal{A}'$ are equal;
- *convertibility equivalent* if the **convertibility relations** of $\mathcal{A}$ and $\mathcal{A}'$ are equal;
- *normal form equivalent* if the **normal form functions** of $\mathcal{A}$ and $\mathcal{A}'$ are equal.

Exercise ●●●○○

1. Show that the future equivalence relation is finer than the convertibility equivalence relation.
2. Show that the future equivalence relation is finer than the normal form equivalence relation.
3. Show that neither the convertibility equivalence relation nor the normal form equivalence relation is finer than the other.

The ARS $\mathcal{A}$ is an *acceleration* of $\mathcal{A}'$ if $\mathcal{A}$ and $\mathcal{A}'$ are normal form equivalent and for any normalizing rewrite sequence u' in $\mathcal{A}'$, there is in $\mathcal{A}$ a rewrite sequence u such that u and u' have the same starting and ending elements, and $\ell \cdot u \leqslant \ell \cdot u'$.

Proposition [Confluence, normal forms, and convertibility]

Let $\mathcal{A} := (X, \Rightarrow)$ be a confluent ARS. For any $x \in X$, if there exists $y \in [x]_\equiv$ such that y is a normal form of $\mathcal{A}$, then y is the unique normal form of $[x]_\equiv$ in $\mathcal{A}$ and $y \in x^{\Rightarrow}$.

Proof. Let us first prove that y is the unique normal form of $[x]_\equiv$ in $\mathcal{A}$. For this, let $y' \in [x]_\equiv$ be a normal form of $\mathcal{A}$. Since $y, y' \in [x]_\equiv$, we have $y \equiv y'$. Moreover, as $\mathcal{A}$ is confluent, by Theorem [Semi-confluence, confluence, and Church-Rosser property] and Proposition [Church-Rosser property], there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$. Since y and y' are normal forms of $\mathcal{A}$, $y = z = y'$. Hence, $y = y'$ as expected.

Let us finally prove that $y \in x^{\Rightarrow}$. Since $x \equiv y$, by Theorem [Semi-confluence, confluence, and Church-Rosser property] and Proposition [Church-Rosser property], there exists $z \in X$ such that $x \Rightarrow^* z$ and $y \Rightarrow^* z$. As y is a normal form of $\mathcal{A}$, $y = z$. Therefore, we have $x \Rightarrow^* y$ as expected.

Proposition [Existence of a unique normal form and confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If for any $x \in X$, $x^{\Rightarrow}$ is a singleton, then $\mathcal{A}$ is confluent.

Proof. Let $x, y, y' \in X$ such that $x \Rightarrow^* y$ and $x \Rightarrow^* y'$. By hypothesis, there exists a unique $z \in X$ such that $z \in y^{\Rightarrow}$ (resp. $z' \in X$ such that $z' \in y'^{\Rightarrow}$). Moreover, since $x \Rightarrow^* y$ and $y \Rightarrow^* z$ (resp. $x \Rightarrow^* y'$ and $y' \Rightarrow^* z'$), we have $x \Rightarrow^* z$ (resp. $x \Rightarrow^* z'$). From the hypothesis, $x^{\Rightarrow}$ is a singleton, implying that $z = z'$. By Proposition [Confluence], this implies that $\mathcal{A}$ is confluent.

Exercise ●●○○○

Prove that the statement obtained from the one of the previous proposition by asking that $\#x^{\Rightarrow} \leqslant 1$ instead of $\#x^{\Rightarrow} = 1$ does not implies that $\mathcal{A}$ is confluent.

Note that the previous exercise consists in showing that the converse of Proposition [Confluence and normal forms] does not hold.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

When $\mathcal{A}$ is both terminating and confluent, $(X, \Rightarrow)$ is *convergent*.

Note that the properties of termination and confluence are **independent**.

Exercise ●○○○○

Define a non-terminating and non-confluent ARS, a terminating and non-confluent ARS, a non-terminating and confluent ARS, and a convergent ARS.

Theorem (Convergence and quotient by the convertibility relation)

Let $\mathcal{A} := (X, \Rightarrow)$ be a convergent ARS. The set of normal forms of $\mathcal{A}$ is a complete set of representatives of the quotient $X/_{\equiv}$.

Proof. By Propositions [Termination and normal forms] and [Confluence and normal forms], each $\equiv$-equivalence class contains exactly one normal form of $\mathcal{A}$. This implies the statement of the Theorem.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

An $x \in X$ is *ambiguous* in $\mathcal{A}$ if there exist $y, y' \in x^{\Rightarrow}$ such that $y \neq y'$.

Theorem [Newman's Lemma]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is terminating and locally confluent, then $\mathcal{A}$ is confluent.

Proof. Assume that $x \in X$ is ambiguous in $\mathcal{A}$ so that there exist $z, z' \in x^{\Rightarrow}$ with $z \neq z'$. Hence, we have $x \Rightarrow y \Rightarrow^* z$ and $x \Rightarrow y' \Rightarrow^* z'$ for some $y, y' \in X$. By the fact that $\mathcal{A}$ is locally confluent and by Proposition [Local confluence], there is $t \in X$ such that $y \Rightarrow^* t$ and $y' \Rightarrow^* t$. By Proposition [Termination and normal forms], $t^{\Rightarrow}$ contains a $u \in X$. Since $z \neq z'$, we have $u \neq z$ or $u \neq z'$ (or both). If $u \neq z$, then as $y \Rightarrow^* z$, $y \Rightarrow^* u$, and z and u are normal forms of $\mathcal{A}$, y is ambiguous. Similarly, If $u \neq z'$, y' is ambiguous. This shows that each ambiguous element of X admits at least one ambiguous successor.

This property implies that there is in $\mathcal{A}$ an infinite rewrite sequence consisting of ambiguous elements. Since $\mathcal{A}$ is terminating, $\mathcal{A}$ cannot have any ambiguous element.

Therefore, for any $x \in X$, by Proposition [Termination and normal forms], $x^{\Rightarrow}$ is a singleton. By Proposition [Existence of a unique normal form and confluence], $\mathcal{A}$ is confluent.

4. Combinatorics of terms

4.1. Terms

Definition

A *graded set* is a pair (X, rk) where X is a set, called the *underlying set*, and $\mathrm{rk} : X \to \mathbb{N}$ is a function, called the *rank function*.

Let $\mathcal{G} := (X, \mathrm{rk})$ be a graded set.

For any $x \in X$, the natural number $\mathrm{rk} \cdot x$ is the *rank* of x in $\mathcal{G}$.

For any $n \in \mathbb{N}$, let

$$\mathcal{G} \cdot n := \{x \in X : \mathrm{rk} \cdot x = n\}.$$

If for any $n \in \mathbb{N}$, $\mathcal{G} \cdot n$ is finite, then $\mathcal{G}$ is *combinatorial*.

A graded set $\mathcal{G}' := (X', \mathrm{rk}')$ is a *sub-graded set* of $\mathcal{G}$ if $X' \subset X$ and rk' is the restriction of rk on the domain X'.

Examples

Let the graded set $\mathcal{G} := (\{a, b\}^*, \ell)$. Since there are finitely many words on $\{a, b\}$ of any length $n \in \mathbb{N}$, $\mathcal{G}$ is combinatorial.

Let the graded set $\mathcal{G}' := (\{a, b\}^*, \ell_a)$. Since $\mathcal{G}' \cdot 0 = \{\epsilon, b, bb, bbb, \dots\}$ is an infinite set, $\mathcal{G}'$ is not combinatorial.

A *signature* $\mathcal{S}$ is a graded set whose elements of the underlying set of a signature are called *constants* and rank function is called *arity function*.

Example

Let the graded set $\mathcal{S}_{\mathbb{N}^2} := (\{\mathsf{c}_{i,j} : i, j \in \mathbb{N}\}, \mathrm{ar})$ where for any $\mathsf{c}_{i,j} \in \mathcal{S}_{\mathbb{N}^2}$, $\mathrm{ar} \cdot \mathsf{c}_{i,j} = i$.

In the sequel, to lighten the notations, we shall write simply c_i for $\mathsf{c}_{i,0}$.

A *set of variables* V is a set whose elements are called *variables*.

Example

Let the set of variables $\mathsf{V}_{\mathbb{N}} := \{\mathsf{v}_i : i \in \mathbb{N}\}$.

Let, for any $n \in \mathbb{N}$, V_n be the subset of $\mathsf{V}_{\mathbb{N}}$ containing only the variables v_i such that $i \in [n]$.

We **always implicitly assume** that any set of variables is **disjoint** from the underlying set of any signature.

Definition

Given a signature $\mathcal{S}$ and a set of variables V, an $\mathcal{S},\mathsf{V}$-*term* is defined recursively to be

- either a variable $v \in \mathsf{V}$;
- or a pair $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ where $c \in \mathcal{S} \cdot n$ for an $n \in \mathbb{N}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The set of $\mathcal{S},\mathsf{V}$-terms is denoted by $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

If $\mathfrak{t} = (c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ is an $\mathcal{S},\mathsf{V}$-term, for any $j \in [n]$, the *j-th subterm* of $\mathfrak{t}$ is the $\mathcal{S},\mathsf{V}$-term $\mathfrak{t} \cdot j := \mathfrak{t}_j$.

Examples

Here are some $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms:

$$\mathsf{v}_8, \quad (\mathsf{c}_0, ()), \quad \mathfrak{t} := (\mathsf{c}_2, ((\mathsf{c}_2, (\mathsf{v}_2, \mathsf{v}_1)), (\mathsf{c}_1, (\mathsf{v}_2))))$$

Moreover, we have $\mathfrak{t} \cdot 1 = (\mathsf{c}_2, (\mathsf{v}_2, \mathsf{v}_1))$ and $\mathfrak{t} \cdot 2 = (\mathsf{c}_1, (\mathsf{v}_2))$.

We have also $\llcorner \mathfrak{t} \cdot 1 \lrcorner \cdot 2 = \mathfrak{t} \cdot 1 \cdot 2 = \mathsf{v}_1$.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

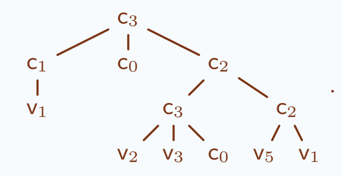
It follows immediately from the definition that $\mathfrak{t}$ is a decorated ordered rooted tree.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := (\mathsf{c}_3, ((\mathsf{c}_1, (\mathsf{v}_1)), (\mathsf{c}_0, ()), (\mathsf{c}_2, ((\mathsf{c}_3, (\mathsf{v}_2, \mathsf{v}_3, (\mathsf{c}_0, ()))), (\mathsf{c}_2, (\mathsf{v}_5, \mathsf{v}_1)))))).$$

This term is represented as the decorated rooted tree



By considering the usual terminology of graphs and trees and by seeing $\mathfrak{t}$ as a tree:

- a *node* of $\mathfrak{t}$ is any vertex of $\mathfrak{t}$. Such nodes are decorated on $\mathsf{V} \sqcup C$;
- an *internal node* of $\mathfrak{t}$ is a node decorated on C;
- a *leaf* of $\mathfrak{t}$ is a node decorated on V.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

The *functional notation* of $\mathfrak{t}$ is obtained by writing $c(\mathfrak{t}_1, \dots, \mathfrak{t}_n)$ instead of $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$.

Example

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of the previous example admits the functional notation

$$\mathsf{c}_3(\mathsf{c}_1(\mathsf{v}_1), \mathsf{c}_0(), \mathsf{c}_2(\mathsf{c}_3(\mathsf{v}_2, \mathsf{v}_3, \mathsf{c}_0()), \mathsf{c}_2(\mathsf{v}_5, \mathsf{v}_1))).$$

The *applicative notation* of $\mathfrak{t}$ is obtained by writing $c\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ instead of $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$.

Example

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of the previous example admits the applicative notation

$$\mathsf{c}_3 \underline{\mathsf{c}_1 \mathsf{v}_1} \mathsf{c}_0 \underline{\mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0} \underline{\mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1}}.$$

In the sequel, we shall **mainly use the applicative notation** for $\mathcal{S}, \mathsf{V}$-terms up to some small exceptions.

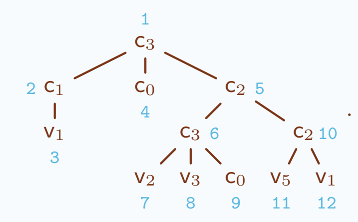
Given a signature $\mathcal{S}$, a set of variables V, and an $\mathcal{S},\mathsf{V}$-term $\mathfrak{t}$, the *preorder traversal* of $\mathfrak{t}$ is defined recursively as follows:

- if $\mathfrak{t} = v$ where $v \in \mathsf{V}$, then the leaf forming $\mathfrak{t}$ is visited;
- otherwise, $\mathfrak{t} = c\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ where $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, and $\mathfrak{t}_1$, ..., $\mathfrak{t}_n$ are $\mathcal{S},\mathsf{V}$-terms. The root of $\mathfrak{t}$ is visited first, and then, $\mathfrak{t}_1$, ..., and $\mathfrak{t}_n$ are visited according to their respective preorder traversals.

This procedure induces a total order on the nodes of $\mathfrak{t}$ where the first visited element is the smallest one.

Example

Here is the $\mathcal{S}_{\mathbb{N}^2},\mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ of the previous example with the indices of its nodes according to their order of appearance in the preorder traversal of $\mathfrak{t}$:



Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

- The *word of* $\mathfrak{t}$ is the word $\mathrm{w} \cdot \mathfrak{t}$ on $\mathsf{V} \cup C$ obtained by reading from left to right the symbols of the applicative notation of $\mathfrak{t}$.
- The *variable word* $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the subword of $\mathrm{w} \cdot \mathfrak{t}$ made of the letters of V.
- The *constant word* $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the subword of $\mathrm{w} \cdot \mathfrak{t}$ made of the letters of C.
- The *variable set* $\mathrm{Vars} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the set $\{\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \cdot i : i \in [\ell \cdot \lfloor \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \rfloor]\}$.

Example

Let $\mathfrak{t}$ be the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of the previous example. Since $\mathsf{c}_3 \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor \mathsf{c}_0 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1 \rfloor \rfloor$, we have

- $\mathrm{w} \cdot \mathfrak{t} = \mathsf{c}_3 \mathsf{c}_1 \mathsf{v}_1 \mathsf{c}_0 \mathsf{c}_2 \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1$;
- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \mathsf{v}_5 \mathsf{v}_1$;
- $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t} = \mathsf{c}_3 \mathsf{c}_1 \mathsf{c}_0 \mathsf{c}_2 \mathsf{c}_3 \mathsf{c}_0 \mathsf{c}_2$;
- $\mathrm{Vars} \cdot \mathfrak{t} = \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_5\}$.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

When

- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \epsilon$, $\mathfrak{t}$ is *ground*;
- $\ell_{\mathsf{v}} \cdot \lfloor \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \rfloor \leqslant 1$ for all $\mathsf{v} \in \mathsf{V}$, $\mathfrak{t}$ is *linear*.

Examples

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_3\mathsf{c}_0\lfloor\mathsf{c}_2\mathsf{c}_0\mathsf{c}_0\rfloor\mathsf{c}_0$ is ground (and thus, also linear);
- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_2\mathsf{v}_1\lfloor\mathsf{c}_2\mathsf{v}_4\mathsf{v}_3\rfloor$ is not ground and is linear;
- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_2\mathsf{v}_3\mathsf{v}_3$ is neither ground nor linear.

Exercise ○○○○○

Give a necessary and sufficient condition on a signature $\mathcal{S}$ for the existence of ground $\mathcal{S}, \mathsf{V}$-terms, where V is any set of variables.

Exercise ○○○○○

Give an example of a signature $\mathcal{S}$ so that the set of linear $\mathcal{S}, \{\mathsf{v}\}$-terms is infinite, where v is a variable.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

Let the following **rank functions** on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$:

- the *length* $\ell \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w} \cdot \mathfrak{t}$;
- the *variable length* $\ell_{\mathrm{var}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t}$. Let also $\ell_v \cdot \mathfrak{t} := \ell_v \cdot \llcorner \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \lrcorner$ be the number of occurrences of the variable $v \in \mathsf{V}$ in $\mathfrak{t}$;
- the *constant length* $\ell_{\mathrm{cns}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t}$. Let also $\ell_c \cdot \mathfrak{t} := \ell_c \cdot \llcorner \mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t} \lrcorner$ be the number of occurrences of the constant $c \in C$ in $\mathfrak{t}$;
- the *height* $\mathrm{ht} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the maximum number of internal nodes on a downward path starting at the root of $\mathfrak{t}$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := \mathsf{c}_3 \llcorner \mathsf{c}_1 \mathsf{v}_1 \lrcorner \mathsf{c}_0 \llcorner \mathsf{c}_2 \llcorner \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \lrcorner \llcorner \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1 \lrcorner \lrcorner.$$

We have $\ell \cdot \mathfrak{t} = 12$, $\ell_{\mathrm{var}} \cdot \mathfrak{t} = 5$, $\ell_{\mathrm{cns}} \cdot \mathfrak{t} = 7$, and $\mathrm{ht} \cdot \mathfrak{t} = 4$.

Besides, $\mathrm{ht} \cdot \mathsf{v}_1 = 0$ and $\mathrm{ht} \cdot \mathsf{c}_0 = 1$.

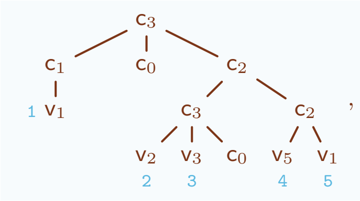
Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

Let us consider the following indexation of nodes, leaves, and internal nodes of $\mathfrak{t}$:

- for any $i \in [\ell \cdot \mathfrak{t}]$, the *$i$-th node* of $\mathfrak{t}$ is the **i-th node** visited according to the preorder traversal of $\mathfrak{t}$;
- for any $i \in [\ell_{\mathrm{var}} \cdot \mathfrak{t}]$, the *$i$-th leaf* of $\mathfrak{t}$ is the **i-th leaf** visited according to the preorder traversal of $\mathfrak{t}$;
- for any $i \in [\ell_{\mathrm{cns}} \cdot \mathfrak{t}]$, the *$i$-th internal node* of $\mathfrak{t}$ is the **i-th internal node** visited according to the preorder traversal of $\mathfrak{t}$.

Example

An $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term with the indices of its leaves (left) and the indices of its internal nodes (right):



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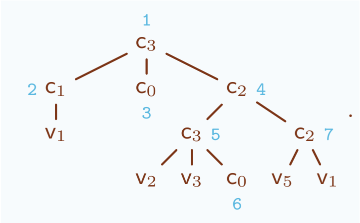

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Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

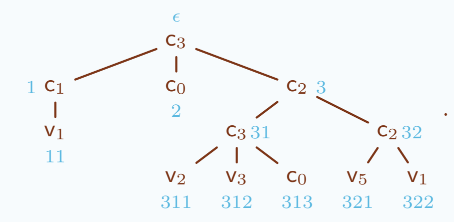
For any word u on $\mathbb{N} \setminus \{0\}$, we define the $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t} \cdot u$ recursively as

$$\mathfrak{t} \cdot u := \begin{cases} \mathfrak{t} & \text{if } u = \epsilon, \\ (\mathfrak{t} \cdot j) \cdot u' & \text{otherwise, where } u = j \,.\, u' \text{ with } j \in \mathbb{N} \setminus \{0\} \text{ and } u' \in (\mathbb{N} \setminus \{0\})^*. \end{cases}$$

The function $u \mapsto \mathfrak{t} \cdot u$ is **partial**. When $\mathfrak{t} \cdot u$ is defined, the $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t} \cdot u$ is the *u-subterm* of $\mathfrak{t}$ and u is the *position* of the root of $\mathfrak{t} \cdot u$ within $\mathfrak{t}$. The set of positions within $\mathfrak{t}$ is denoted by $\mathrm{P} \cdot \mathfrak{t}$.

Example

Here is an $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ with the positions of its nodes:



We have

$$\mathrm{P} \cdot \mathfrak{t} = \{\epsilon, 1, 11, 2, 3, 31, 311, 312, 313, 32, 321, 322\}.$$

Definition

Given a signature $\mathcal{S}$, a *labeled $\mathcal{S}$-term* is an $\mathcal{S}, \mathbb{N} \setminus \{0\}$-term.

Let $\mathcal{S}$ be a signature and $\mathfrak{t}$ be a labeled $\mathcal{S}$-term.

The *variable rank* $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is 0 if $\mathrm{Vars} \cdot \mathfrak{t} = \emptyset$ and, otherwise, is the maximal element among the variables in $\mathrm{Vars} \cdot \mathfrak{t}$.

Assume that $\mathfrak{t}$ contains n leaves. When

- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \lfloor \theta \cdot 1 \rfloor \dots \lfloor \theta \cdot n \rfloor$, where $\theta : [n] \to [m]$ is a surjective function for a certain $m \in \mathbb{N}$, $\mathfrak{t}$ is *packed*;
- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \lfloor \sigma \cdot 1 \rfloor \dots \lfloor \sigma \cdot n \rfloor$, where $\sigma \in \mathfrak{S}_n$, $\mathfrak{t}$ is *standard*;
- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = 1 \dots n$, $\mathfrak{t}$ is *planar*.

Note that a planar labeled $\mathcal{S}$-term is standard and that a standard labeled $\mathcal{S}$-term is packed.

Examples

- Let $\mathfrak{t} := \mathsf{c}_3 \, 2 \, \lfloor \mathsf{c}_1 \, 1 \rfloor \, \lfloor \mathsf{c}_2 \, 1 \, 4 \rfloor$. We have $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t} = 4$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}$ is not packed.
- Let $\mathfrak{t}' := \mathsf{c}_2 \, 2 \, \lfloor \mathsf{c}_2 \, 2 \, 1 \rfloor$. We have $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}' = 2$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}'$ is packed but not standard.
- Let $\mathfrak{t}'' := \mathsf{c}_3 \, 2 \, 1 \, 3$. We have $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}'' = 3$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}''$ is standard but not planar.
- Let $\mathfrak{t}''' := \mathsf{c}_3 \, 1 \, 2 \, 3$. We have $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}''' = 3$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}'''$ is planar.

Exercise ○○○○○

Classify the following sequences of symbols depending on whether they form valid applicative notations of some $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms:

- ☐ v_1;
- ☐ c_1;
- ☐ c_0;
- ☐ $\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1$;
- ☐ $\mathsf{c}_2 \mathsf{v}_2 \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor$;
- ☐ $\mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor \lfloor \mathsf{c}_0 \mathsf{v}_1 \rfloor$;
- ☐ $\mathsf{c}_3 \lfloor \mathsf{v}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor \mathsf{v}_2 \lfloor \mathsf{c}_1 \mathsf{v}_4 \rfloor$;
- ☐ $\mathsf{c}_3 \lfloor \mathsf{c}_1 \mathsf{c}_0 \mathsf{v}_1 \rfloor \mathsf{v}_2 \lfloor \mathsf{c}_1 \mathsf{v}_4 \rfloor$;
- ☐ $\mathsf{c}_3 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor \mathsf{v}_2 \lfloor \mathsf{c}_1 \mathsf{v}_4 \rfloor$.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}$-term. Give a necessary and sufficient condition for the fact that the word of $\mathfrak{t}$ is a valid applicative notation of an $\mathcal{S}, \mathsf{V}$-term.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables.

1. Provide a recursive definition of the height of $\mathcal{S}, \mathsf{V}$-terms.
2. Show that for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, $\mathrm{ht} \cdot \mathfrak{t} \leqslant \ell_{\mathrm{cns}} \cdot \mathfrak{t}$.

Exercise ●●○○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and X be a subset of $(\mathbb{N} \setminus \{0\})^*$. Give a necessary and sufficient condition for the fact that there exists $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathrm{P} \cdot \mathfrak{t} = X$.

Exercise ●●○○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term. Prove that for any $i \in [\ell \cdot \mathfrak{t}]$, the position u of the i-th node of $\mathfrak{t}$ appears at i-th position among $\mathrm{P} \cdot \mathfrak{t}$ when seen as a sorted list w.r.t. the lexicographic order on $(\mathbb{N} \setminus \{0\})^*$.

Exercise ●●○○○○

Given a monoid $(\mathcal{M}, \star, \mathbb{1})$ and a set X, a partial function $\theta : X \to \mathcal{M} \to X$ is a *partial right monoid action* of $\mathcal{M}$ on X if for any $x \in X$, $\theta \cdot x \cdot \mathbb{1} = x$, and for any $x \in X$ and $m_1, m_2 \in \mathcal{M}$, $\theta \cdot \llcorner \theta \cdot x \cdot m_1 \lrcorner \cdot m_2$ is defined iff $\theta \cdot x \cdot \llcorner m_1 \star m_2 \lrcorner$ is defined, and when they are defined, these two elements of X are equal.

Prove that for any signature $\mathcal{S}$ and any set of variables V, the function $\theta : \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \to (\mathbb{N} \setminus \{0\})^* \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ defined for any $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ and $u \in (\mathbb{N} \setminus \{0\})^*$ by $\theta \cdot \mathfrak{t} \cdot u := \mathfrak{t} \cdot u$ is a partial right monoid action of the free monoid $((\mathbb{N} \setminus \{0\})^*, \centerdot, \epsilon)$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

Exercise ●○○○○

Let, for any $n \in \mathbb{N}$, X_n be the set of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is ground, any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \dots\}$, and $\ell_{\mathsf{c}_0} \cdot \mathfrak{t} = n$.
Let, for any $n \in \mathbb{N}$, X'_n be the set of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is planar, any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \dots\}$, and $\mathrm{rk}_{\mathsf{v}} \cdot \mathfrak{t} = n$.

Describe a bijection $\theta : X_n \to X'_n$.

Exercise ●●○○○

Give a necessary and sufficient condition on a signature $\mathcal{S}$ for the fact that the set of packed labeled $\mathcal{S}$-terms of variable rank n is finite for any $n \in \mathbb{N}$.

Exercise ●●●○○

Provide necessary and sufficient conditions on a signature $\mathcal{S}$ and a set of variables V so that

1. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell)$ is combinatorial;
2. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell_{\mathrm{var}})$ is combinatorial;
3. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell_{\mathrm{cns}})$ is combinatorial;
4. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \mathrm{ht})$ is combinatorial.

4.2. Substitutions

Definition

Given a signature $\mathcal{S}$ and a set of variables V, an $\mathcal{S},\mathsf{V}$-*substitution* is a function $\mathsf{V} \to \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

Let $\mathcal{S}$ be a signature, V be a set of variables, and σ be an $\mathcal{S},\mathsf{V}$-substitution. The *domain* of σ is the set

$$\mathrm{Dom}{\cdot}\sigma := \{v \in \mathsf{V} : \sigma{\cdot}v \neq v\}.$$

Any subset S of $\mathsf{V} \times \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ such that $(v, \mathfrak{t}) \in S$ and $(v, \mathfrak{t}') \in S$ implies $\mathfrak{t} = \mathfrak{t}'$ *specifies* the $\mathcal{S},\mathsf{V}$-substitution $[S]$ defined, for any $v \in \mathsf{V}$, by

$$[S]{\cdot}v := \begin{cases} \mathfrak{t} & \text{if } (v, \mathfrak{t}) \in S, \\ v & \text{otherwise.} \end{cases}$$

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution

$$\sigma := [\{(\mathsf{v}_2, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1), (\mathsf{v}_3, \mathsf{v}_3), (\mathsf{v}_5, \mathsf{v}_6)\}].$$

We have $\mathrm{Dom}{\cdot}\sigma = \{\mathsf{v}_2, \mathsf{v}_5\}$, $\sigma{\cdot}\mathsf{v}_2 = \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1$, $\sigma{\cdot}\mathsf{v}_5 = \mathsf{v}_6$, and $\sigma{\cdot}\mathsf{v}_i = \mathsf{v}_i$ for all $i \in \mathbb{N} \setminus \{2, 5\}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Given an $\mathcal{S}, \mathsf{V}$-substitution σ, the *extension* of σ is the function $\overline{\sigma} : \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V} \to \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ defined recursively, for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, by

$$\overline{\sigma}{\cdot}\mathfrak{t} := \begin{cases} \sigma{\cdot}v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ c \llcorner \overline{\sigma}{\cdot}\mathfrak{t}_1 \lrcorner \dots \llcorner \overline{\sigma}{\cdot}\mathfrak{t}_n \lrcorner & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \dots \mathfrak{t}_n. \end{cases}$$

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_2\, \mathsf{v}_1\, \mathsf{v}_2), (\mathsf{v}_2, \mathsf{c}_1\, \mathsf{v}_5), (\mathsf{v}_3, \mathsf{v}_4)\}]$. We have

$$\overline{\sigma}{\cdot}\mathsf{c}_3\, \mathsf{v}_2 \llcorner \mathsf{c}_2\, \mathsf{v}_3\, \mathsf{v}_2 \lrcorner \llcorner \mathsf{c}_1\, \mathsf{c}_0 \lrcorner = \mathsf{c}_3 \llcorner \mathsf{c}_1\, \mathsf{v}_5 \lrcorner \llcorner \mathsf{c}_2\, \mathsf{v}_4 \llcorner \mathsf{c}_1\, \mathsf{v}_5 \lrcorner \lrcorner \llcorner \mathsf{c}_1\, \mathsf{c}_0 \lrcorner.$$

An $\mathcal{S}, \mathsf{V}$-substitution σ is a *renaming* if σ is injective and for any $v \in \mathsf{V}$, $\sigma{\cdot}v = v'$ for a $v' \in \mathsf{V}$.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [\{(\mathsf{v}_1, \mathsf{v}_1), (\mathsf{v}_2, \mathsf{v}_3), (\mathsf{v}_3, \mathsf{v}_4), (\mathsf{v}_4, \mathsf{v}_2)\}]$. This $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution is a renaming and we have

$$\overline{\sigma}{\cdot}\mathsf{c}_3\, \mathsf{v}_2 \llcorner \mathsf{c}_2\, \mathsf{v}_3\, \mathsf{v}_2 \lrcorner \llcorner \mathsf{c}_1\, \mathsf{c}_0 \lrcorner = \mathsf{c}_3\, \mathsf{v}_3 \llcorner \mathsf{c}_2\, \mathsf{v}_4\, \mathsf{v}_3 \lrcorner \llcorner \mathsf{c}_1\, \mathsf{c}_0 \lrcorner.$$

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *disjoint union* of two $\mathcal{S}, \mathsf{V}$-substitutions σ_1 and σ_2 such that $\mathrm{Dom} \cdot \sigma_1 \cap \mathrm{Dom} \cdot \sigma_2 = \emptyset$ is the $\mathcal{S}, \mathsf{V}$-substitution $\sigma_1 \sqcup \sigma_2$ defined, for any $v \in \mathsf{V}$, by

$$\lfloor \sigma_1 \sqcup \sigma_2 \rfloor \cdot v := \begin{cases} \sigma_1 \cdot v & \text{if } v \in \mathrm{Dom} \cdot \sigma_1, \\ \sigma_2 \cdot v & \text{otherwise.} \end{cases}$$

The *composition* of two $\mathcal{S}, \mathsf{V}$-substitutions σ_1 and σ_2 is the $\mathcal{S}, \mathsf{V}$-substitution $\sigma_1 \circ \sigma_2$ defined, for any $v \in \mathsf{V}$, by

$$\lfloor \sigma_1 \circ \sigma_2 \rfloor \cdot v := \overline{\sigma_1} \cdot \lfloor \sigma_2 \cdot v \rfloor.$$

Example

We have the following composition of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions:

$$[\{(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2)\}] \circ [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_1), (\mathsf{v}_2, \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3)\}] = [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_3 \mathsf{c}_0 \mathsf{v}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2)\}].$$

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term, $v_1 \dots v_n$ be a sequence of pairwise distinct variables of V, $n \in \mathbb{N}$, and $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ be a sequence of $\mathcal{S}, \mathsf{V}$-terms. The *composition of* $\mathfrak{t}$ *and* $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ *on* $v_1 \dots v_n$ is the $\mathcal{S}, \mathsf{V}$-term

$$\mathfrak{t}[\{(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)\}] := \overline{[\{(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)\}]} \cdot \mathfrak{t}.$$

The $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}[(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)]$ is built by **replacing simultaneously** each occurrence of a variable $v_i \in \mathsf{V}$ of $\mathfrak{t}$ by $\mathfrak{t}_i$ for all $i \in [n]$.

Examples

We have the following compositions of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms:

- $\mathsf{c}_3\ \mathsf{v}_2 \lfloor \mathsf{c}_2\ \mathsf{v}_1\ \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor [\{(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_1\ \mathsf{v}_5), (\mathsf{v}_4, \mathsf{v}_1)\}] = \mathsf{c}_3 \lfloor \mathsf{c}_1\ \mathsf{v}_5 \rfloor \lfloor \mathsf{c}_2\ \mathsf{c}_0 \lfloor \mathsf{c}_1\ \mathsf{v}_5 \rfloor \rfloor \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor$;
- $\mathsf{c}_2\ \mathsf{v}_1\ \mathsf{v}_3[\{(\mathsf{v}_1, \mathsf{v}_2), (\mathsf{v}_2, \mathsf{v}_2), (\mathsf{v}_3, \mathsf{v}_2), (\mathsf{v}_4, \mathsf{v}_2), (\mathsf{v}_5, \mathsf{v}_2)\}] = \mathsf{c}_2\ \mathsf{v}_2\ \mathsf{v}_2$.

Let $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ and $v \in \mathsf{V}$. The *partial composition of* $\mathfrak{t}$ *and* $\mathfrak{t}'$ *on* v is the $\mathcal{S}, \mathsf{V}$-term

$$\mathfrak{t} \curvearrowleft_v \mathfrak{t}' := \mathfrak{t}[\{(v, \mathfrak{t}')\}].$$

Let $\mathcal{S}$ be a signature.

A *labeled $\mathcal{S}$-substitution* is an $\mathcal{S}, \mathbb{N} \setminus \{0\}$-substitution.

The *labeled $\mathcal{S}$-substitution* of a sequence $\mathfrak{t}_1 \dots \mathfrak{t}_n$ of labeled $\mathcal{S}$-terms is the labeled $\mathcal{S}$-substitution

$$[\mathfrak{t}_1, \dots, \mathfrak{t}_n] := [\{(1, \mathfrak{t}_1), \dots, (n, \mathfrak{t}_n)\}].$$

Example

Let the labeled $\mathcal{S}_{\mathbb{N}^2}$-substitution σ defined by

$$\sigma := [2, 2, \mathsf{c}_2\, 3\, \mathsf{c}_0, \mathsf{c}_0, 5].$$

We have $\sigma \cdot 1 = 2$, $\sigma \cdot 2 = 2$, $\sigma \cdot 3 = \mathsf{c}_2\, 3\, \mathsf{c}_0$, $\sigma \cdot 4 = \mathsf{c}_0$, and for any $i \geqslant 5$, $\sigma \cdot i = i$.
Therefore, $\mathrm{Dom} \cdot \sigma = \{1, 3, 4\}$.

Note that, by using the previous notations, in this context of labeled $\mathcal{S}$-substitutions, for any labeled $\mathcal{S}$-term $\mathfrak{t}$ and any sequence $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$, $n \in \mathbb{N}$, of labeled $\mathcal{S}$-terms,

$$\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n] = \mathfrak{t}[\{(1, \mathfrak{t}'_1), \dots, (n, \mathfrak{t}'_n)\}].$$

The composition of labeled $\mathcal{S}$-substitutions, when $\mathcal{S}$ is a signature, can be described in the following way.

Proposition [Composition of labeled substitutions]

For any signature $\mathcal{S}$, any sequences $\mathfrak{t}_1 \dots \mathfrak{t}_n$, $n \in \mathbb{N}$, and $\mathfrak{s}_1 \dots \mathfrak{s}_m$, $m \in \mathbb{N}$, of labeled $\mathcal{S}$-terms, and any labeled $\mathcal{S}$-term $\mathfrak{r}$ such that $m \geqslant \mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{r}$,

$$\overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n] \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_m]} \cdot \mathfrak{r} = \overline{\left[\overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n]} \cdot \mathfrak{s}_1, \dots, \overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n]} \cdot \mathfrak{s}_m\right]} \cdot \mathfrak{r}.$$

Exercise ●●○○○

Prove that the condition about the variable rank of $\mathfrak{r}$ is necessary in the previous proposition.

Exercise ●●●○○

Prove the previous proposition.

Theorem [Relations of compositions of labeled terms]

For any signature $\mathcal{S}$, the following relations hold:

- for any $n \in \mathbb{N}$, $i \in [n]$, and any sequence $\mathrm{t}_1 \dots \mathrm{t}_n$ of labeled $\mathcal{S}$-terms,
$$i\left[\mathrm{t}_1, \dots, \mathrm{t}_n\right] = \mathrm{t}_i;$$
- for any $n \in \mathbb{N}$ and any labeled $\mathcal{S}$-term t such that $n \geqslant \mathrm{rk_v} \cdot \mathrm{t}$,
$$\mathrm{t}[1, \dots, n] = \mathrm{t};$$
- for any $n, m \in \mathbb{N}$, any labeled $\mathcal{S}$-term t, and any sequences $\mathrm{t}'_1 \dots \mathrm{t}'_n$ and $\mathrm{t}''_1 \dots \mathrm{t}''_m$ of labeled $\mathcal{S}$-terms such that $n \geqslant \mathrm{rk_v} \cdot \mathrm{t}$, and $m \geqslant \mathrm{rk_v} \cdot \mathrm{t}'_i$ for all $i \in [n]$,
$$\underbrace{\mathrm{t}[\mathrm{t}'_1, \dots, \mathrm{t}'_n]}\left[\mathrm{t}''_1, \dots, \mathrm{t}''_m\right] = \mathrm{t}\left[\mathrm{t}'_1\left[\mathrm{t}''_1, \dots, \mathrm{t}''_m\right], \dots, \mathrm{t}'_n\left[\mathrm{t}''_1, \dots, \mathrm{t}''_m\right]\right].$$

Proof. The first two relations are immediate. The third one is a consequence of Proposition [Composition of labeled substitutions].

Exercise ○○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Exhibit the neutral element w.r.t. the composition of $\mathcal{S}, \mathsf{V}$-substitutions.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Show that the composition of $\mathcal{S}, \mathsf{V}$-substitutions is not commutative.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Show that the composition of $\mathcal{S}, \mathsf{V}$-substitutions is associative.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and let $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms. Describe a necessary and sufficient condition on both $\mathfrak{t}$ and $\mathfrak{t}'$ for the existence of two $\mathcal{S}, \mathsf{V}$-substitutions σ and σ' satisfying $\overline{\sigma} \cdot \mathfrak{t} = \mathfrak{t}'$ and $\overline{\sigma'} \cdot \mathfrak{t}' = \mathfrak{t}$.

5. Term series

5.1. Formal series

Let $\mathbb{K}$ be a field.

For any set X, let $\mathbb{K}\langle X \rangle$ be the **$\mathbb{K}$-linear span** of X. By definition, X is a basis of $\mathbb{K}\langle X \rangle$.

Each element of $\mathbb{K}\langle X \rangle$ is a $\mathbb{K}, X$*-polynomial*.

Example

By setting $A := \{a, b\}$, the $\mathbb{K}$-linear combination

$$ab - ba + 2baa$$

is a $\mathbb{K}, A^*$-polynomial.

A $\mathbb{K}, X$-polynomial f is a **$\mathbb{K}$-linear combination** of elements of X so that f can be written as a **finite sum**

$$f = \sum_{x \in X} \lambda_x \, x,$$

where each $\lambda_x \in \mathbb{K}$, $x \in X$, is the *coefficient* of x in f.

The number of $x \in X$ such that $\lambda_x \neq 0$ is **finite**.

Definition

For any field $\mathbb{K}$ and set X, the *space of $\mathbb{K}, X$-series* is the $\mathbb{K}$-vector space $\mathbb{K}\langle\langle X\rangle\rangle$ defined as the dual space of $\mathbb{K}\langle X\rangle$.

Let $\mathbb{K}$ be a field and X be a set.

A $\mathbb{K}, X$-series is a **linear form** $\mathbf{f} : \mathbb{K}\langle X\rangle \to \mathbb{K}$ whose value on $x \in X$ is the *coefficient* $\mathbf{f} \cdot x$ of x in $\mathbf{f}$.

The *support* of $\mathbf{f} \in \mathbb{K}\langle\langle X\rangle\rangle$ is the set

$$\mathrm{Supp} \cdot \mathbf{f} := \{x \in X : \mathbf{f} \cdot x \neq 0\}.$$

The *canonical pairing* of a $\mathbb{K}, X$-polynomial f and a $\mathbb{K}, X$-series $\mathbf{f}$ is the element of $\mathbb{K}$ defined by

$$\langle f, \mathbf{f}\rangle := \mathbf{f} \cdot f = \sum_{x \in X} \lfloor f \cdot x \rfloor \lfloor \mathbf{f} \cdot x \rfloor.$$

Given $X' \subseteq X$, the *characteristic series of X'* is the $\mathbb{K}, X$-series $[X']$ defined, for any $x \in X$, by

$$[X'] \cdot x := [x \in X'].$$

For any $x \in X$, when there is no confusion, we denote by x the $\mathbb{K}, X$-series $[\{x\}]$.

Note in particular that, for any $x \in X$ and $\mathbf{f} \in \mathbb{K}\langle\langle X\rangle\rangle$, $\langle x, \mathbf{f}\rangle$ is the coefficient $\mathbf{f} \cdot x$ of x in $\mathbf{f}$.

Let $\mathbb{K}$ be a field and X be a set.

The *infinite sum notation* of $\mathbf{f} \in \mathbb{K}\langle\langle X \rangle\rangle$ allows us to formally write

$$\mathbf{f} = \sum_{x \in X} \langle x, \mathbf{f} \rangle \, x.$$

Examples

Let the alphabet $A := \{a, b\}$. Here are some $\mathbb{K}, A^*$-series:

- ☐ 0;
- ☐ aba;
- ☐ $ab + 2baa$;
- ☐ $1 + a + aa + aaa + \cdots$;
- ☐ $[A^*]$;
- ☐ $\sum_{w \in A^*} \lfloor \ell \cdot w \rfloor \, w$;
- ☐ $\sum_{w \in A^*} \lfloor \ell_a \cdot w \rfloor \, w$;
- ☐ $\sum_{w_1, w_2 \in A^*} w_1 \bullet w_2$.

Exercise ●○○○○

Express the coefficients $\langle w, \mathbf{f} \rangle$ for all $w \in A^*$ such that $\ell \cdot w \leqslant 3$ for all $\mathbb{K}, A^*$-series $\mathbf{f}$ among the examples above.

Let $\mathbb{K}$ be a field.

For any alphabet A,

- $\mathbb{K}\langle\langle A^*\rangle\rangle$ is the $\mathbb{K}$-vector space of *noncommutative formal power series on* A;

  Examples

  The previous examples show noncommutative formal power series on A with $A = \{a, b\}$.

- $\mathbb{K}\langle\langle \mathrm{Mon} \cdot A\rangle\rangle$ is the $\mathbb{K}$-vector space of *formal power series on* A, where $\mathrm{Mon} \cdot A$ is the set of *monomials over* A that are **commutative words** on A.

  Examples

  Let $A = \{a, b\}$. In $\mathbb{K}\langle\langle \mathrm{Mon} \cdot A\rangle\rangle$, we have

$$\sum_{w \in A^*} w = \epsilon + a + b + aa + ab + ba + bb + aaa + aab + aba + abb + baa + bab + bba + bbb + \cdots$$

$$= \epsilon + a + b + aa + 2ab + bb + aaa + 3aab + 3abb + bbb + \cdots .$$

The linear function $\phi : \mathbb{K}\langle\langle A^*\rangle\rangle \to \mathbb{K}\langle\langle \mathrm{Mon} \cdot A\rangle\rangle$ satisfying, for any $w \in A^*$, $\phi \cdot w = \overline{w}$ where $\overline{w}$ is the **commutative image** of w, extends uniquely to a linear function between $\mathbb{K}\langle\langle A^*\rangle\rangle$ and $\mathbb{K}\langle\langle \mathrm{Mon} \cdot A\rangle\rangle$.

Let $\mathbb{K}$ be a field, X be a set, and $\mathrm{rk}_1 \dots \mathrm{rk}_k$, $k \geqslant 1$, be a sequence of functions such that for any $i \in [k]$, (X, rk_i) is a **graded set**.

Let the **alphabet of variables** $\mathsf{Z} := \{\mathsf{z}_i : i \in \mathbb{N} \setminus \{0\}\}$. To lighten the notation, we sometimes denote by z the variable z_1.

For any $\mathbf{f} \in \mathbb{K}\langle\langle X \rangle\rangle$, the $\mathrm{rk}_1 \dots \mathrm{rk}_k$-*trace* of $\mathbf{f}$ is the $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{Z}$-series

$$\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k} {\cdot} \mathbf{f} := \sum_{x \in X} \langle x, \mathbf{f} \rangle \, \mathsf{z}_1^{\mathrm{rk}_1 \cdot x} \dots \mathsf{z}_k^{\mathrm{rk}_k \cdot x}.$$

Observe that the function $\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k}$ is **linear** and that $\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k} {\cdot} \mathbf{f}$ **may not be well-defined**.

Example

Let the $\mathbb{K}, \mathfrak{S}$-series

$$\mathbf{f} := \sum_{\sigma \in \mathfrak{S}} (-1)^{\mathrm{inv} \cdot \sigma} \sigma = \epsilon + 1 + 12 - 21 + 123 - 132 - 213 + 231 + 312 - 321 + \cdots.$$

By defining $\mathrm{des}{\cdot}\sigma$ as the number of descents of $\sigma \in \mathfrak{S}$, we have

$$\mathrm{tr}_{\ell \, \mathrm{des}} {\cdot} \mathbf{f} = \mathsf{z}_1^0 \mathsf{z}_2^0 + \mathsf{z}_1^1 \mathsf{z}_2^0 + \mathsf{z}_1^2 \mathsf{z}_2^0 + (-1) \mathsf{z}_1^2 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^0 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^1 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^1 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^2 + \cdots.$$

$$= 1 + \mathsf{z}_1 + \mathsf{z}_1^2 - \mathsf{z}_1^2 \mathsf{z}_2 + \mathsf{z}_1^3 - \mathsf{z}_1^3 \mathsf{z}_2^2 + \cdots.$$

Let $\mathbb{K}$ be a field and $\mathcal{G} := (X, \mathrm{rk})$ be a combinatorial graded set.

The *generating series of $\mathcal{G}$* is the $\mathbb{K}, \mathrm{Mon} \cdot \{\mathsf{z}\}$-series

$$\langle \mathcal{G} \rangle := \mathrm{tr}_{\mathrm{rk}} \cdot [X].$$

Example

Consider the graded set $\mathcal{G} := (\{a, b\}^*, \ell)$. Since

$$[\{a, b\}^*] = \epsilon + a + b + aa + ab + ba + bb + aaa + aab + aba + abb + baa + bab + bba + bbb + \cdots,$$

we have

$$\langle \mathcal{G} \rangle = 1 + 2\mathsf{z} + 4\mathsf{z}^2 + 8\mathsf{z}^3 + \cdots.$$

Note that when $\mathcal{G}$ is not combinatorial, $\langle \mathcal{G} \rangle$ **is not well-defined**.

Exercise ●○○○○

Provide an example of a graded set (X, rk) such that $\langle (X, \mathrm{rk}) \rangle$ is not well-defined. Introduce another rank function rk' such that $\langle (X, \mathrm{rk}') \rangle$ is well-defined.

5.2. Products on series

For any set Y and $n \in \mathbb{N}$, an *n-operation on Y* is a function of type

$$\underbrace{Y \to \cdots \to Y}_{n \text{ times}} \to Y.$$

Note that a 0-operation is a **constant**.

Given a **2-operation** θ on a set Y, for any $k \geqslant 1$, let $\theta^{(k)}$ be the k-operation defined recursively, for any $y_1, \dots, y_k \in Y$, by

$$\theta^{(k)} \cdot y_1 \cdot \cdots \cdot y_k := \begin{cases} y_1 & \text{if } k = 1, \\ \theta \cdot \underline{\theta^{(k-1)} \cdot y_1 \cdot \cdots \cdot y_{k-1}} \cdot y_k & \text{otherwise.} \end{cases}$$

Example

Let the alphabet $A := \{a, b\}$ and let the 2-operation θ on A^* satisfying $\theta \cdot w_1 \cdot w_2 = w_1 \centerdot w_2$ for any $w_1, w_2 \in A^*$.

This 2-operation θ has type $A^* \to A^* \to A^*$.

The 3-operation $\theta^{(3)}$ has type $A^* \to A^* \to A^* \to A^*$ and satisfies, for any $w_1, w_2, w_3 \in A^*$,

$$\theta^{(3)} \cdot w_1 \cdot w_2 \cdot w_3 = \theta \cdot \underline{\theta \cdot w_1 \cdot w_2} \cdot w_3 = (w_1 \centerdot w_2) \centerdot w_3 = w_1 \centerdot w_2 \centerdot w_3.$$

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X.

The *extension of θ on* $\mathbb{K}\langle\langle X\rangle\rangle$ is the n-operation $\bar{\theta}$ on $\mathbb{K}\langle\langle X\rangle\rangle$ defined, for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$, by

$$\bar{\theta} \cdot \mathbf{f}_1 \cdot \dots \cdot \mathbf{f}_n := \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \dots \cdot x_n.$$

In other words, $\bar{\theta}$ is the **multilinear function induced by** θ.

Note that $\bar{\theta} \cdot \mathbf{f}_1 \cdot \dots \cdot \mathbf{f}_n$ **may not be well-defined**.

Example

Consider the alphabet A and the 2-operation θ on A^* of the previous example.

We have

$$\bar{\theta} \cdot \llcorner a + bb \lrcorner \cdot \llcorner aa + 2b + ba \lrcorner = aaa + 2ab + aba + bbaa + 2bbb + bbba,$$

and

$$\bar{\theta} \cdot \llcorner \sum_{w_1 \in A^*} w_1 \lrcorner \cdot \llcorner \sum_{w_2 \in A^*} w_2 \lrcorner = \sum_{w_1, w_2 \in A^*} \theta \cdot w_1 \cdot w_2 = \sum_{w_1, w_2 \in A^*} w_1 \bullet w_2 = \epsilon + 2a + 2b + 3aa + 3ab + 3ba + 3bb + \cdots.$$

Exercise ●○○○○

Let $\mathbb{K}$ be a field and X be a set. For any $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let π_i be the n-operation on X such that for any $x_1, \dots, x_n \in X$, $\pi_i \cdot x_1 \cdot \cdots \cdot x_n := x_i$. Provide a description of $\overline{\pi_n} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n$ for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X \rangle\rangle$.

Exercise ●○○○○

Give an explicit example of a 2-operation θ on a set X such that $\overline{\theta} \cdot \mathbf{f}_1 \cdot \mathbf{f}_2$ is not well-defined for some $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{K}\langle\langle X \rangle\rangle$ where $\mathbb{K}$ is a field.

Exercise ●●○○○

Let the alphabet $A := \{a, b\}$. Let the 2-operation θ satisfying $\theta \cdot w_1 \cdot w_2 = w_1 \centerdot w_2$ for any $w_1, w_2 \in A^*$. Give, for any $w \in A^*$, an explicit expression for the coefficient

$$\left\langle w, \overline{\theta} \cdot \underbrace{\sum_{w_1 \in A^*} \lfloor \ell_a \cdot w_1 \rfloor\, w_1} \cdot \underbrace{\sum_{w_2 \in A^*} \lfloor \ell_b \cdot w_2 \rfloor\, w_2} \right\rangle.$$

Proposition [Multilinearity of extensions of operations]

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X. If the extension $\bar{\theta}$ of θ on $\mathbb{K}\langle\langle X\rangle\rangle$ is well-defined, then $\bar{\theta}$ is multilinear.

Proof. Let $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$, $\alpha \in \mathbb{K}$, $i \in [n]$, and $\mathbf{f}'_i \in \mathbb{K}\langle\langle X\rangle\rangle$. By multilinearity (in particular in its second argument) of the canonical pairing, we have

$$\bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \underbrace{\alpha \mathbf{f}_i + \mathbf{f}'_i} \cdot \ \dots \ \cdot \mathbf{f}_n = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \langle x_i, \alpha \mathbf{f}_i + \mathbf{f}'_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \alpha \langle x_i, \mathbf{f}_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n + \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \langle x_i, \mathbf{f}'_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n$$

$$= \alpha \, \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \mathbf{f}_i \cdot \ \dots \ \cdot \mathbf{f}_n + \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \mathbf{f}'_i \cdot \ \dots \ \cdot \mathbf{f}_n.$$

This shows the linearity of $\bar{\theta}$ in its i-th argument.

Proposition [Coefficients in series with extensions of operations]

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X. If the extension $\bar{\theta}$ of θ on $\mathbb{K}\langle\langle X\rangle\rangle$ is well-defined, then for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$ and $x \in X$,

$$\left\langle x, \bar{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \right\rangle = \sum_{(x_1, \dots, x_n) \in X^n} [x = \theta \cdot x_1 \cdot \cdots \cdot x_n] \prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle.$$

Proof. By multilinearity (in particular in its second argument) of the canonical pairing, we have

$$\left\langle x, \bar{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \right\rangle = \left\langle x, \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \cdots \cdot x_n \right\rangle$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \langle x, \theta \cdot x_1 \cdot \cdots \cdot x_n \rangle$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) [x = \theta \cdot x_1 \cdot \cdots \cdot x_n].$$

This shows the statement.

Let $\mathcal{G} := (X, \mathrm{rk})$ be a graded set. An n-operation θ is rk-*graded* if, for any $x_1, \dots, x_n \in X$,

$$\mathrm{rk} \cdot \llcorner \theta \cdot x_1 \cdot \cdots \cdot x_n \lrcorner = \mathrm{rk} \cdot x_1 + \cdots + \mathrm{rk} \cdot x_n.$$

Examples

Let A be an alphabet.

- The concatenation operation $\centerdot$ on A^* is an ℓ-graded 2-operation. Indeed, for any $w_1, w_2 \in A^*$, $\ell \cdot \llcorner w_1 \centerdot w_2 \lrcorner = \ell \cdot w_1 + \ell \cdot w_2$.
- The reversal function $w \mapsto w^{\mathrm{r}}$ on A^* where, for any $w \in A^*$ and $i \in [\ell \cdot w]$, $w^{\mathrm{r}} \cdot i := w \cdot \llcorner \ell \cdot w - i + 1 \lrcorner$ is an ℓ-graded 1-operation.
- The 1-operation $\theta := w \mapsto w_{|A'}$ on A^* where, for any $w \in A^*$ and $A' \subseteq A$, $w_{|A'}$ is the subword of w consisting of its letters belonging to A', is not ℓ-graded. For instance, for $A := \{a, b, c\}$, $A' := \{a, b\}$, and $w := abca$, we have $\ell \cdot \llcorner \theta \cdot w \lrcorner = \ell \cdot aba = 3 \neq 4 = \ell \cdot abca = \ell \cdot w$.

  Nevertheless, θ is ℓ_x-graded for any $x \in A'$.

If θ is an rk-graded 2-operation, then for any $k \geqslant 1$, then $\theta^{(k)}$ is also rk-graded (follows by induction on k).

Proposition [Graded operations and traces]

Let $\mathbb{K}$ be a field, (X, rk) be a combinatorial graded set, and θ be an rk-graded n-operation on X. If the extension $\bar{\theta}$ of θ on $\mathbb{K}\langle\langle X\rangle\rangle$ is well-defined, then

$$\mathrm{tr}_{\mathrm{rk}} \cdot \lfloor \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \cdots \ \cdot \mathbf{f}_n \rfloor = \prod_{i \in [n]} \mathrm{tr}_{\mathrm{rk}} \cdot \mathbf{f}_i$$

where $\prod$ is the iterated product on monomials.

Proof. By definition of $\bar{\theta}$ and $\mathrm{tr}_{\mathrm{rk}}$, by linearity of $\mathrm{tr}_{\mathrm{rk}}$, and since θ is rk-graded, we have

$$\mathrm{tr}_{\mathrm{rk}} \cdot \lfloor \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \cdots \ \cdot \mathbf{f}_n \rfloor = \mathrm{tr}_{\mathrm{rk}} \cdot \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathrm{tr}_{\mathrm{rk}} \cdot \lfloor \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n \rfloor = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathsf{z}^{\mathrm{rk} \cdot \lfloor \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n \rfloor}$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathsf{z}^{\mathrm{rk} \cdot x_1 + \cdots + \mathrm{rk} \cdot x_n} = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \, \mathsf{z}^{\mathrm{rk} \cdot x_i} \right)$$

$$= \prod_{i \in [n]} \left(\sum_{x \in X} \langle x, \mathbf{f}_i \rangle \, \mathsf{z}^{\mathrm{rk} \cdot x} \right) = \prod_{i \in [n]} \mathrm{tr}_{\mathrm{rk}} \cdot \mathbf{f}_i.$$

5.3. Enumeration

The *integer sequence* of a **combinatorial** graded set $\mathcal{G} := (X, \mathrm{rk})$ is the sequence $(\#\cdot\lfloor\mathcal{G}\cdot n\rfloor)_{n \in \mathbb{N}}$.

An approach to **compute the entries of the integer sequence** of $\mathcal{G}$ consists in providing a description of the **generating series** $\langle\mathcal{G}\rangle$ of $\mathcal{G}$.

Here are the steps:

1. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$.
2. Consider a collection $\{\theta_i : i \in [k]\}$ of rk-graded n_i-operation on X, where $n_i \in \mathbb{N}$, $i \in [k]$, $k \in \mathbb{N}$.
3. Express the characteristic series $[X]$ of X via a system of equations using the extensions $\overline{\theta_i}$ on $\mathbb{K}\langle\langle X\rangle\rangle$ of the θ_i, $i \in [k]$.
4. By applying the rk-trace function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the equations of the system, transform the previous system of equations of series of $\mathbb{K}\langle\langle X\rangle\rangle$ into a system of equations on formal power series of $\mathbb{K}\langle\langle \mathrm{Mon}\cdot\mathsf{Z}\rangle\rangle$.
5. Use the previous equations to express, for any $n \in \mathbb{N}$, the coefficient $\langle \mathsf{z}^n, \langle\mathcal{G}\rangle\rangle$.

It follows from Proposition [Graded operations and traces] that for any $n \in \mathbb{N}$, $\langle \mathsf{z}^n, \langle\mathcal{G}\rangle\rangle = \#\cdot\lfloor\mathcal{G}\cdot n\rfloor$ as expected.

Let us give some **examples** fitting in this framework, involving the enumeration of some families of paths.

A *path* is a nonempty word on $\mathbb{N}$. The *rank* $\mathrm{rk}{\cdot}p$ of a path p is $\ell{\cdot}p - 1$.

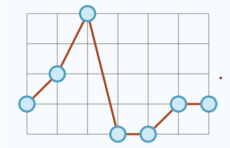
Let P be the graded set whose underlying set is the set of paths and rank function is rk.

A path p of rank $n-1$, $n \geqslant 1$, is drawn as the **set of points** $\{(i-1, p{\cdot}i) : i \in [n]\}$, where any pair of adjacent points is connected by a **step**.

Note that the **rank** of a path p is the **number of steps** of p.

Example

The path 1240011 has rank 6 and is depicted as

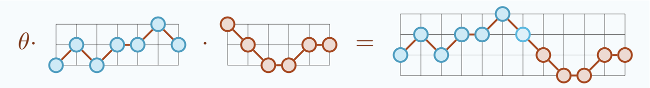

.

The *concatenation* of paths is the 2-operation θ defined for any $p_1 \in \mathsf{P} \cdot \llcorner n_1 - 1 \lrcorner$, $n_1 \geqslant 1$, and $p_2 \in \mathsf{P} \cdot \llcorner n_2 - 1 \lrcorner$, $n_2 \geqslant 1$, by

$$\theta \cdot p_1 \cdot p_2 := \begin{cases} \llcorner \uparrow_k \cdot p_1' \lrcorner \centerdot p_2 & \text{if } p_1 \cdot n_1 \leqslant p_2 \cdot 1, \\ p_1 \centerdot \llcorner \uparrow_k \cdot p_2' \lrcorner & \text{otherwise,} \end{cases}$$

where for any $w \in \mathbb{N}^*$ and $j \in \mathbb{N}$, $\uparrow_j \cdot w$ is the word obtained by incrementing by j each letter of w, $k := |p_1 \cdot n_1 - p_2 \cdot 1|$, and p_1' (resp. p_2') is the word obtained by deleting the last (resp. first) letter of p_1 (resp. p_2).

Example



Exercise ●○○○○

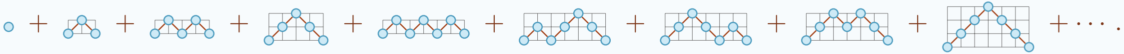
Prove that the 2-operation θ on paths is rk-graded.

Exercise ●●●○○

Prove that the 2-operation θ on paths is associative.

Let DP be the graded set of *Dyck paths*, defined as the sub-graded set of P containing the paths whose first and last letters are 0, and obtained by iterated concatenation via θ of the paths 01 and 10.

By denoting by X the underlying set of DP, we have

$[X] =$ 

Lemma [Decomposition of Dyck paths]

If p is a Dyck path, then

1. either $p = \circ$;
2. or there exists a unique pair (p_1, p_2) of Dyck paths such that

$p = \theta^{(4)} \cdot$  $\cdot p_1 \cdot$  $\cdot p_2 .$

Exercise ●●○○○

Prove the previous lemma.

From Lemma [Decomposition of Dyck paths], it follows that the characteristic series of DP satisfies the equation in $\mathbb{K}\langle\langle \mathsf{P} \rangle\rangle$

$$[X] = \circ + \overline{\theta^{(4)}} \cdot$$  $$\cdot [X] \cdot$$  $$\cdot [X].$$

By considering the images by the function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the previous equation, and since θ is rk-graded,

$$\langle \mathsf{DP} \rangle = 1 + \mathsf{z}^2 \langle \mathsf{DP} \rangle^2.$$

From this, we obtain that for any $n \in \mathbb{N}$,

$$\langle \mathsf{z}^n, \langle \mathsf{DP} \rangle \rangle = \begin{cases} 1 & \text{if } n = 0, \\ \sum_{n_1, n_2 \in \mathbb{N}} [n_1 + n_2 = n - 2] \langle \mathsf{z}^{n_1}, \langle \mathsf{DP} \rangle \rangle \langle \mathsf{z}^{n_2}, \langle \mathsf{DP} \rangle \rangle & \text{otherwise.} \end{cases}$$

By computing the first values, we obtain that the integer sequence of DP starts by

$$1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, 429, 0.$$

By deleting the zeros, this is the Catalan integer sequence (A000108).

Let MP be the graded set of *Motzkin paths*, defined as the sub-graded set of P containing the paths whose first and last letters are 0, and obtained by iterated concatenation via θ of the paths 01, 10, and 00.

By denoting by X the underlying set of MP, we have

$$[X] =$$ 

Lemma [Decomposition of Motzkin paths]

If p is a Motzkin path, then

1. either $p = \circ$;
2. or there exists a unique Motzkin path p_1 such that $p = \theta \cdot$  $\cdot p_1$;
3. or there exists a unique pair (p_1, p_2) of Motzkin paths such that $p = \theta^{(4)} \cdot$  $\cdot p_1 \cdot$  $\cdot p_2$.

Exercise ●●○○○

Prove the previous lemma.

From Lemma [Decomposition of Motzkin paths], it follows that the characteristic series of MP satisfies the equation in $\mathbb{K}\langle\langle \mathsf{P} \rangle\rangle$

$$[X] = \circ + \bar{\theta} \cdot \circ\!\!-\!\!\circ \cdot [X] + \overline{\theta^{(4)}} \cdot \nearrow \cdot [X] \cdot \searrow \cdot [X].$$

By considering the images by the function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the previous equation, and since θ is rk-graded,

$$\langle \mathsf{MP} \rangle = 1 + \mathsf{z} \langle \mathsf{MP} \rangle + \mathsf{z}^2 \langle \mathsf{MP} \rangle^2.$$

From this, we obtain that for any $n \in \mathbb{N}$,

$$\langle \mathsf{z}^n, \langle \mathsf{MP} \rangle \rangle = \begin{cases} 1 & \text{if } n = 0, \\ \langle \mathsf{z}^{n-1}, \langle \mathsf{MP} \rangle \rangle + \sum\limits_{n_1, n_2 \in \mathbb{N}} [n_1 + n_2 = n - 2] \langle \mathsf{z}^{n_1}, \langle \mathsf{MP} \rangle \rangle \langle \mathsf{z}^{n_2}, \langle \mathsf{MP} \rangle \rangle & \text{otherwise.} \end{cases}$$

By computing the first values, we obtain that the integer sequence of MP starts by

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572.$$

This is the sequence of **Motzkin numbers** (A001006).

5.4. Term series and substitutions

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term and $v_1 \dots v_n$, $n \in \mathbb{N}$, be a sequence of pairwise distinct variables of V.

The $\mathfrak{t}, v_1 \dots v_n$*-grafting operation* is the n-operation $\theta_{\mathfrak{t}, v_1 \dots v_n}$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$ defined, for any $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, by

$$\theta_{\mathfrak{t}, \mathsf{v}_1 \dots \mathsf{v}_n} \cdot \mathfrak{t}'_1 \cdot \dots \cdot \mathfrak{t}'_n := \mathfrak{t}[\{(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)\}].$$

Example

By considering the signature $\mathcal{S}_{\mathbb{N}^2}$, the set of variables $\mathrm{V}_{\mathbb{N}}$, and by setting $\mathfrak{t} := \mathsf{c}_2 \llcorner \mathsf{c}_1 \mathsf{v}_3 \lrcorner \llcorner \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \lrcorner$, we have

$$\theta_{\mathfrak{t}, \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \cdot \llcorner \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_2 \mathsf{v}_4 \lrcorner \cdot \llcorner \mathsf{c}_1 \mathsf{v}_1 \lrcorner \cdot \llcorner \mathsf{c}_2 \mathsf{v}_2 \llcorner \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_6 \lrcorner \lrcorner = \mathsf{c}_2 \llcorner \mathsf{c}_1 \llcorner \mathsf{c}_2 \mathsf{v}_2 \llcorner \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_6 \lrcorner \lrcorner \lrcorner \llcorner \mathsf{c}_2 \llcorner \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_2 \mathsf{v}_4 \lrcorner \llcorner \mathsf{c}_1 \mathsf{v}_1 \lrcorner \lrcorner.$$

If $\mathbb{K}$ is a field, an $\mathcal{S}, \mathrm{V}$*-term series* is a $\mathbb{K}, \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$-series.

Such $\mathfrak{t}$-grafting operations, together with $\mathcal{S}, \mathrm{V}$-term series, can be used to describe characteristic series of some families of $\mathcal{S}, \mathrm{V}$-terms in order to enumerate them w.r.t. some adequate rank functions.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term, $v_1 \dots v_n$, $n \in \mathbb{N}$, be a sequence of pairwise distinct variables of V.

By setting $V := \{v_1, \dots, v_n\}$, for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$,

- for any $v \in \mathsf{V}$,

$$\ell_v \cdot \underbrace{\theta_{\mathfrak{t}, v_1 \dots v_n} \cdot \mathfrak{t}'_1 \cdot \cdots \cdot \mathfrak{t}'_n} = [v \notin V] \underbrace{\ell_v \cdot \mathfrak{t}} + \sum_{i \in [n]} \underbrace{\ell_{v_i} \cdot \mathfrak{t}} \underbrace{\ell_v \cdot \mathfrak{t}'_i}.$$

  When $\mathrm{Vars} \cdot \mathfrak{t} = V$ and $\mathfrak{t}$ is linear, the n-operation $\theta_{\mathfrak{t}, v_1 \dots v_n}$ is ℓ_v-**graded**. In this case, $\theta_{\mathfrak{t}, v_1 \dots v_n}$ is also ℓ_{var}-**graded**;
- for any $c \in C$,

$$\ell_c \cdot \underbrace{\theta_{\mathfrak{t}, v_1 \dots v_n} \cdot \mathfrak{t}'_1 \cdot \cdots \cdot \mathfrak{t}'_n} = \ell_c \cdot \mathfrak{t} + \sum_{i \in [n]} \underbrace{\ell_{v_i} \cdot \mathfrak{t}} \underbrace{\ell_c \cdot \mathfrak{t}'_i}.$$

  When $\mathrm{Vars} \cdot \mathfrak{t} = V$, $\mathfrak{t}$ is linear, and $\ell_c \cdot \mathfrak{t} = 0$, the n-operation $\theta_{\mathfrak{t}, v_1 \dots v_n}$ is ℓ_c-**graded**.

Exercise ●●●○○

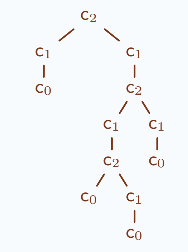
Let $\mathcal{S}$ be a signature and V be a set of variables. Provide a necessary and sufficient condition for a rank function rk, an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, and a sequence $v_1 \dots v_n$, $n \in \mathbb{N}$, of pairwise distinct variables of V for the fact that the n-operation $\theta_{\mathfrak{t}, v_1 \dots v_n}$ is a rk-graded.

An *alternating unary binary tree* is an $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-term $\mathfrak{t}$ such that

1. any internal node of $\mathfrak{t}$ is decorated on $\{\mathtt{c}_0, \mathtt{c}_1, \mathtt{c}_2\}$;
2. for any positions u and $u \cdot j$ in $\mathfrak{t}$ such that $j \in \mathbb{N} \setminus \{0\}$, the decorations of the internal nodes at positions u and $u \cdot j$ in $\mathfrak{t}$ are different.

Example

Here is an alternating unary binary tree:



Let AUB be the graded set whose underlying set is the set of alternating unary binary trees and rank function is $\ell_{\mathtt{c}_0}$. Recall that this function sends each alternating unary binary tree $\mathfrak{t}$ to the number of internal nodes of $\mathfrak{t}$ decorated by $\mathtt{c}_0$.

Exercise ●●○○○

Prove that the graded set AUB is combinatorial.

The 1-*grafting* of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms is the 1-operation $\theta_1 := \theta_{\mathsf{c}_1 \mathsf{v}_1, \mathsf{v}_1}$.

Similarly, the 2-*grafting* of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms is the 2-operation $\theta_2 := \theta_{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2, \mathsf{v}_1 \mathsf{v}_2}$.

Examples

On alternating unary binary trees, we have

- $\theta_1 \cdot \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \rfloor = \mathsf{c}_1 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \rfloor$;
- $\theta_2 \cdot \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \cdot \lfloor \mathsf{c}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{c}_0 \rfloor \rfloor = \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{c}_0 \rfloor \rfloor$.

Exercise ●○○○○

Prove that the 1-operation θ_1 and the 2-operation θ_2 on $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms are ℓ_{c_0}-graded.

Exercise ●○○○○

Prove that the 2-operation θ_2 on $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}$-terms is not associative.

By denoting by X the underlying set of AUB, we have

$$[X] = \mathsf{c}_0 + \mathsf{c}_1\mathsf{c}_0 + \mathsf{c}_2\mathsf{c}_0\mathsf{c}_0 + \mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor + \mathsf{c}_2\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\mathsf{c}_0 + \mathsf{c}_2\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor + \mathsf{c}_1\lfloor\mathsf{c}_2\mathsf{c}_0\mathsf{c}_0\rfloor + \mathsf{c}_1\lfloor\mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\rfloor$$
$$+\mathsf{c}_1\lfloor\mathsf{c}_2\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\mathsf{c}_0\rfloor + \mathsf{c}_1\lfloor\mathsf{c}_2\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\rfloor + \mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\lfloor\mathsf{c}_2\mathsf{c}_0\mathsf{c}_0\rfloor\rfloor + \mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\lfloor\mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\mathsf{c}_0\rfloor\rfloor\rfloor + \cdots.$$

Lemma [Decomposition of alternating unary binary trees]

If $\mathfrak{t}$ is an alternating unary binary tree, then

1. either $\mathfrak{t} = \mathsf{c}_0$;
2. or there exists a unique alternating unary binary tree $\mathfrak{t}_1$ having root decorated by c_0 or by c_2 such that $\mathfrak{t} = \theta_1 \cdot \mathfrak{t}_1$;
3. or there exists a unique pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of alternating unary binary trees having roots decorated by c_0 or by c_1 such that $\mathfrak{t} = \theta_2 \cdot \mathfrak{t}_1 \cdot \mathfrak{t}_2$.

Exercise ●○○○○

Prove the previous lemma.

From Lemma [Decomposition of alternating unary binary trees], it follows that the characteristic series of AUB satisfies the following system of equations in $\mathbb{K}\langle\langle \mathfrak{T} \cdot \mathcal{S}_{\mathbb{N}^2} \cdot \mathsf{V}_{\mathbb{N}} \rangle\rangle$:

$$[X] = \mathsf{c}_0 + \overline{\theta_1} \cdot [X_2] + \overline{\theta_2} \cdot [X_1] \cdot [X_1],$$

$$[X_1] = \mathsf{c}_0 + \overline{\theta_1} \cdot [X_2],$$

$$[X_2] = \mathsf{c}_0 + \overline{\theta_2} \cdot [X_1] \cdot [X_1],$$

where X_1 is the subset of X consisting of alternating unary binary trees whose roots are decorated by c_0 or by c_1, and where X_2 is the subset of X consisting of alternating unary binary trees whose roots are decorated by c_0 or by c_2.

By considering the images by the function $\mathrm{tr}_{\ell_{\mathsf{c}_0}}$ on both sides of the previous equations of the system, and since θ_1 and θ_2 are ℓ_{c_0}-graded,

$$\langle \mathsf{AUB} \rangle = \mathsf{z} + \mathbf{f}_2 + \mathbf{f}_1^2,$$

$$\mathbf{f}_1 = \mathsf{z} + \mathbf{f}_2,$$

$$\mathbf{f}_2 = \mathsf{z} + \mathbf{f}_1^2,$$

where $\mathbf{f}_1 := \mathrm{tr}_{\ell_{\mathsf{c}_0}} \cdot [X_1]$ and $\mathbf{f}_2 := \mathrm{tr}_{\ell_{\mathsf{c}_0}} \cdot [X_2]$.

From the previous system of equations, we obtain

$$\langle \mathsf{AUB} \rangle = 2\mathbf{f}_2 \qquad \text{and} \qquad \mathbf{f}_2 = \mathrm{z} + \mathrm{z}^2 + 2\mathrm{z}\mathbf{f}_2 + \mathbf{f}_2^2$$

so that, for any $n \in \mathbb{N}$,

$$\langle \mathrm{z}^n, \langle \mathsf{AUB} \rangle \rangle = 2\langle \mathrm{z}^n, \mathbf{f}_2 \rangle \qquad \text{and} \qquad \langle \mathrm{z}^n, \mathbf{f}_2 \rangle = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 2\left\langle \mathrm{z}^{n-1}, \mathbf{f}_2 \right\rangle + \sum\limits_{n_1, n_2 \in [n-1]} [n_1 + n_2 = n] \langle \mathrm{z}^{n_1}, \mathbf{f}_2 \rangle \langle \mathrm{z}^{n_2}, \mathbf{f}_2 \rangle & \text{otherwise.} \end{cases}$$

By computing the first values, we obtain that the integer sequence of AUB starts by

$$0, 2, 8, 32, 160, 896, 5376, 33792, 219648, 1464320, 9957376, 68796416.$$

Exercise ●●●○○

Give a very simple **combinatorial argument** showing that for any $n \geqslant 2$, $\langle \mathrm{z}^n, \langle \mathsf{AUB} \rangle \rangle = 2^{n+1} \mathrm{cat} \cdot \lfloor n - 1 \rfloor$, where, for any $k \in \mathbb{N}$, $\mathrm{cat} \cdot k = \binom{2k}{k} \frac{1}{k+1}$ is the number of binary trees with k internal nodes.

Exercise ●●●○○

Let X be the set of $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-terms $\mathfrak{t}$ such that any internal node of $\mathfrak{t}$ is decorated on $\{\mathtt{c}_0, \mathtt{c}_2, \mathtt{c}_3\}$ and any internal decorated by $\mathtt{c}_3$ has no child which is an internal node decorated by $\mathtt{c}_2$.

1. Prove that the graded set $(X, \ell_{\mathtt{c}_0})$ is combinatorial.
2. Define some grafting operations allowing us to decompose any $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-term of X in a recursive way.
3. Provide a system of equations for $[X]$ using the extensions of the previous grafting operations.
4. Deduce from this system of equations a system of equations for the generating series $\langle (X, \ell_{\mathtt{c}_0}) \rangle$ of $(X, \ell_{\mathtt{c}_0})$.
5. Deduce from this system of equations a recurrence formula to compute the terms of the integer sequence of $(X, \ell_{\mathtt{c}_0})$.

6. Term rewrite systems

6.1. Matchings

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let t and t' be $\mathcal{S}, \mathrm{V}$-terms. If there exists an $\mathcal{S}, \mathrm{V}$-substitution σ such that $\mathrm{t}' = \overline{\sigma} \cdot \mathrm{t}$, then

- t is a *prefix* of t'. This property is denoted by $\mathrm{t} \preccurlyeq_{\mathrm{p}} \mathrm{t}'$;
- the $\mathcal{S}, \mathrm{V}$-substitution σ is a *matching* of t into t'.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term

$$\mathrm{t}' := \mathrm{c}_3 \mathrm{c}_0 \lfloor \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_3 \rfloor \lfloor \mathrm{c}_1 \mathrm{v}_1 \rfloor.$$

- Let $\mathrm{t} := \mathrm{c}_3 \mathrm{v}_1 \mathrm{v}_2 \mathrm{v}_3$. Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathrm{v}_1, \mathrm{c}_0), (\mathrm{v}_2, \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_3), (\mathrm{v}_3, \mathrm{c}_1 \mathrm{v}_1)]$. Since $\mathrm{t}' = \overline{\sigma} \cdot \mathrm{t}$, σ is a matching of t into t' and $\mathrm{t} \preccurlyeq_{\mathrm{p}} \mathrm{t}'$.
- Let $\mathrm{t} := \mathrm{c}_3 \mathrm{v}_1 \mathrm{v}_2 \lfloor \mathrm{c}_1 \mathrm{v}_3 \rfloor$. Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathrm{v}_1, \mathrm{c}_0), (\mathrm{v}_2, \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_3), (\mathrm{v}_3, \mathrm{v}_1)]$. Since $\mathrm{t}' = \overline{\sigma} \cdot \mathrm{t}$, σ is a matching of t into t' and $\mathrm{t} \preccurlyeq_{\mathrm{p}} \mathrm{t}'$.
- Let $\mathrm{t} := \mathrm{c}_3 \mathrm{v}_1 \mathrm{v}_2 \mathrm{v}_2$. By assuming that a $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-substitution σ exists such that σ is a matching of t into t', we would have both $\sigma \cdot \mathrm{v}_2 = \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_3$ and $\sigma \cdot \mathrm{v}_2 = \mathrm{c}_1 \mathrm{v}_1$. This is absurd, so that t is not a prefix of t'.
- Let $\mathrm{t} := \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_2$. By assuming that a $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-substitution σ exists, we would have $\mathrm{t}' = \overline{\sigma} \cdot \mathrm{t}$, so that $\mathrm{c}_3 = \mathrm{t}' \cdot \epsilon = \overline{\sigma} \cdot \mathrm{t} \cdot \epsilon = \mathrm{c}_2$. This is absurd, so that t is not a prefix of t'.

Exercise ●○○○○

Consider the two $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t' := c_2 \lfloor c_1 v_1 \rfloor \lfloor c_2 v_2 v_3 \rfloor$ and $t := c_2 v_1 \lfloor c_2 v_2 v_3 \rfloor$. Give two different matchings of t into t'.

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-term $t' := c_3 \lfloor c_2 v_1 v_2 \rfloor v_2 \lfloor c_2 \lfloor c_1 v_3 \rfloor c_0 \rfloor$. Give examples of $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms t such that t are prefixes of t' of constant lengths n for all $n \in [\![5]\!]$.

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-term $t := c_2 \lfloor c_2 v_1 v_2 \rfloor \lfloor c_2 v_2 v_1 \rfloor$. Give an example of a ground $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-term t' such that $\ell_{\mathrm{cns}} \cdot t' = 9$ and $t \preccurlyeq_{\mathrm{p}} t'$.

Exercise ●○○○○

Define two different $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms t and t' satisfying $t \preccurlyeq_{\mathrm{p}} t'$ and $t' \preccurlyeq_{\mathrm{p}} t$.

A binary relation $\mathcal{R}$ on a set X is a *preorder* if $\mathcal{R}$ is reflexive and transitive.

When, additionally, $\mathcal{R}$ is antisymmetric, $\mathcal{R}$ is an *order relation*.

Proposition [Prefix relation on $\mathcal{S}, \mathsf{V}$-terms]

For any signature $\mathcal{S}$ and set of variables V, the binary relation $\preccurlyeq_{\mathrm{p}}$ is

- a preorder, when seen as a binary relation on $\mathcal{S}, \mathsf{V}$-terms;
- an order relation, when seen as a binary relation on planar labeled $\mathcal{S}$-terms.

Exercise ●○○○○

Give a simple combinatorial characterization of the fact that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$ holds, where $\mathfrak{t}$ and $\mathfrak{t}'$ are two planar labeled $\mathcal{S}$-terms and $\mathcal{S}$ is a signature.

Exercise ●●○○○

Prove the previous proposition.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S},\mathsf{V}$-terms.

A **matching** of $\mathfrak{t}$ into $\mathfrak{t}'$ can be **computed through the ARS** $\mathsf{Matching}_{\mathcal{S},\mathsf{V}} := \left(\{\mathsf{Fail}\} \cup \mathcal{P} \cdot \underline{\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}}^2, \Rightarrow\right)$ such that

$$\{(c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c\,\mathfrak{t}'_1 \dots \mathfrak{t}'_n)\} \sqcup S \Rightarrow \{(\mathfrak{t}_1, \mathfrak{t}'_1), \dots, (\mathfrak{t}_n, \mathfrak{t}'_n)\} \cup S,$$

$$\{(c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c'\,\mathfrak{t}'_1 \dots \mathfrak{t}'_{n'})\} \sqcup S \Rightarrow \mathsf{Fail}, \quad \text{if } c \neq c',$$

$$\{(c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, v)\} \sqcup S \Rightarrow \mathsf{Fail},$$

$$\{(v, \mathfrak{t}), (v, \mathfrak{t}')\} \sqcup S \Rightarrow \mathsf{Fail}, \quad \text{if } v \in \mathsf{V} \text{ and } \mathfrak{t} \neq \mathfrak{t}'.$$

This ARS is used by computing the **normal form** of $\{(\mathfrak{t}, \mathfrak{t}')\}$ and, when this normal form is a set S, by considering the **$\mathcal{S},\mathsf{V}$-substitution** $[S]$ **specified** by S.

This $\mathcal{S},\mathsf{V}$-substitution satisfies $\mathfrak{t}' = \overline{[S]} \cdot \mathfrak{t}$.

The normal form Fail is reached in the case where $\mathfrak{t}$ is not a prefix of $\mathfrak{t}'$.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_3 v_1 \lfloor c_2 v_2 v_3 \rfloor v_1$ and $t' := c_3 \lfloor c_1 v_2 \rfloor \lfloor c_2 \lfloor c_1 v_2 \rfloor v_3 \rfloor \lfloor c_1 v_2 \rfloor$. In $\mathrm{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_1 v_2), (c_2 v_2 v_3, c_2 \lfloor c_1 v_2 \rfloor v_3), (v_1, c_1 v_2)\} \Rightarrow \{(v_1, c_1 v_2), (v_2, c_1 v_2), (v_3, v_3), (v_1, c_1 v_2)\}.$$

We can check that the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-substitution $[\{(v_1, c_1 v_2), (v_2, c_1 v_2), (v_3, v_3)\}]$ is a matching of t into t'.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_2 v_1 \lfloor c_2 v_2 v_3 \rfloor$ and $t' := c_2 c_0 \lfloor c_3 v_1 v_2 v_3 \rfloor$. In $\mathrm{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_0), (c_2 v_2 v_3, c_3 v_1 v_2 v_3)\} \Rightarrow \mathrm{Fail}.$$

We can check that t is not a prefix of t'.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms $t := c_3 v_1 v_2 v_2$ and $t' := c_3 c_0 \lfloor c_1 c_0 \rfloor \lfloor c_1 \lfloor c_1 c_0 \rfloor \rfloor$. In $\mathrm{Matching}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$,

$$\left\{\left(t, t'\right)\right\} \Rightarrow \{(v_1, c_0), (v_2, c_1 c_0), (v_2, c_1 \lfloor c_1 c_0 \rfloor)\} \Rightarrow \mathrm{Fail}.$$

We can check that t is not a prefix of t'.

Due to the following results, the ARS $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$, where $\mathcal{S}$ is any signature and V is any set of variables, computes exactly what is described before.

Theorem [Convergence of $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$]

For any signature $\mathcal{S}$ and set of variables V, the ARS $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$ is convergent.

Proposition [Computation of $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$, and $S := \{(\mathfrak{t}, \mathfrak{t}')\}$.

- If $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$, then the normal form of S in $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$ is the set S' such that $[S']$ is a matching of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Otherwise, the normal form of S in $\mathsf{Matching}_{\mathcal{S},\mathsf{V}}$ is Fail.

6.2. Rewrite mechanics

Definition

A *term rewrite system* (*TRS*) is a triple $(\mathcal{S}, \mathsf{V}, \to)$ where $\mathcal{S}$ is a signature, called the *underlying signature*, V is a set of variables, called the *underlying set of variables*, and $\to$ is a binary relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}$, called the *elementary rewrite relation*.

The elementary rewrite relation must satisfy the following two conditions:

1. for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \to \mathfrak{t}'$ implies that $\mathrm{Vars}\cdot\mathfrak{t}' \subseteq \mathrm{Vars}\cdot\mathfrak{t}$;
2. for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \to \mathfrak{t}'$ implies that $\ell_{\mathrm{cns}}\cdot\mathfrak{t} \geqslant 1$.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS.

If $\mathfrak{t}$ and $\mathfrak{t}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t} \to \mathfrak{t}'$, then

- the pair $(\mathfrak{t}, \mathfrak{t}')$ is a *rewrite rule* of $\mathcal{T}$;
- $\mathfrak{t}$ is the *left-hand side* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$;
- $\mathfrak{t}'$ is the *right-hand side* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$;
- $\mathrm{w}_{\mathrm{cns}}\cdot\mathfrak{t}\cdot 1$ is the *head constant* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$. By 2., this is well-defined;
- for any $\mathcal{S}, \mathsf{V}$-substitution σ, the pair $(\overline{\sigma}\cdot\mathfrak{t}, \overline{\sigma}\cdot\mathfrak{t}')$ is an *instance* of the rewrite rule $(\mathfrak{t}, \mathfrak{t}')$.

Example

Let $\mathsf{Assoc} := ((\{m\}, \mathrm{ar}), \{1, 2, 3\}, \to)$ be the TRS such that $\mathrm{ar}{\cdot}m = 2$ and $m\llcorner\underline{m12}\lrcorner 3 \to m1\llcorner\underline{m23}\lrcorner$.

Example

Let $\mathsf{Assoc}_2 := ((\{m, m'\}, \mathrm{ar}), \{1, 2, 3\}, \to)$ be the TRS such that $\mathrm{ar}{\cdot}m = 2$, $\mathrm{ar}{\cdot}m' = 2$, $m\llcorner\underline{m12}\lrcorner 3 \to m1\llcorner\underline{m23}\lrcorner$, and $m'1\llcorner\underline{m'23}\lrcorner \to m'\llcorner\underline{m'12}\lrcorner 3$.

Example

Let $\mathsf{Eq} := ((\{t, e\}, \mathrm{ar}), \{1\}, \to)$ be the TRS such that $\mathrm{ar}{\cdot}t = 0$, $\mathrm{ar}{\cdot}e = 2$, and $e11 \to t$.

Example

Let $\mathsf{NatAdd} := ((\{z, s, a\}, \mathrm{ar}), \{1, 2\}, \to)$ be the TRS such that $\mathrm{ar}{\cdot}z = 0$, $\mathrm{ar}{\cdot}s = 1$, $\mathrm{ar}{\cdot}a = 2$, $a1z \to 1$, and $a1\llcorner\underline{s2}\lrcorner \to s\llcorner\underline{a12}\lrcorner$.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS.

The *rewrite relation* of $\mathcal{T}$ is the **smallest binary relation** $\Rightarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ satisfying the two following properties:

1. for any instance $(\mathfrak{s}, \mathfrak{s}')$ of a rewrite rule of $\mathcal{T}$,
$$\mathfrak{s} \Rightarrow \mathfrak{s}';$$
2. for any $c \in \mathcal{S}{\cdot}n$, $n \geqslant 1$, and any $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$, $i \in [n]$, such that $\mathfrak{t}_i \Rightarrow \mathfrak{t}'_i$,
$$c\,\mathfrak{t}_1 \dots \mathfrak{t}_i \dots \mathfrak{t}_n \Rightarrow c\,\mathfrak{t}_1 \dots \mathfrak{t}'_i \dots \mathfrak{t}_n.$$

The TRS $\mathcal{T}$ defines the ARS $(\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}, \Rightarrow)$, called *the ARS of* $\mathcal{T}$.

We use the previous notations and notions in the context of ARSs, here on the ARS of $\mathcal{T}$. For instance, $\mathfrak{t} \mapsto \mathfrak{t}^{\Rightarrow^*}$ is the **future function** of $\mathcal{T}$ and $\equiv$ is the **convertibility relation** of $\mathcal{T}$.

Similarly, we write that $\mathcal{T}$ **satisfies a property** P for the fact that the ARS of $\mathcal{T}$ satisfies P. For instance, writing that $\mathcal{T}$ is confluent means that the ARS of $\mathcal{T}$ is confluent.

Example

In Assoc_2, by setting $\mathfrak{t} := m' \lfloor m \lfloor m12 \rfloor 3 \rfloor \lfloor m' 4 \lfloor m' 56 \rfloor \rfloor$, we have

- $\mathfrak{t} \Rightarrow m' \lfloor m' \lfloor m \lfloor m12 \rfloor 3 \rfloor 4 \rfloor \lfloor m' 56 \rfloor$;
- $\mathfrak{t} \Rightarrow m' \lfloor m \lfloor m12 \rfloor 3 \rfloor \lfloor m' \lfloor m' 45 \rfloor 6 \rfloor$;
- $\mathfrak{t} \Rightarrow m' \lfloor m1 \lfloor m23 \rfloor \rfloor \lfloor m' 4 \lfloor m' 56 \rfloor \rfloor$.

Exercise ●○○○○

Give a normalizing rewrite sequence in NatAdd starting from $a \lfloor sz \rfloor \lfloor s \lfloor sz \rfloor \rfloor$.

Exercise ●●○○○

Draw the rewrite graph of the closed sub-ARS of the ARS of Assoc generated by $\{m \lfloor m \lfloor m \lfloor m12 \rfloor 3 \rfloor 4 \rfloor 5\}$.

Exercise ●●○○○

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS. Prove that for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, for any $\mathfrak{t}' \in \mathfrak{t}^{\Rightarrow^*}$, $\mathrm{Vars} \cdot \mathfrak{t}' \subseteq \mathrm{Vars} \cdot \mathfrak{t}$. From this property, prove that all $\mathcal{S}, \mathsf{V}$-terms of the future of any ground $\mathcal{S}, \mathsf{V}$-term are ground.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathrm{V}$-terms. If there exists a position w within $\mathfrak{t}'$ such that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}' \cdot w$, then

- $\mathfrak{t}$ is a *factor* of $\mathfrak{t}'$. This property is denoted by $\mathfrak{t} \preccurlyeq_{\mathrm{f}} \mathfrak{t}'$;
- the word w is an *occurrence of* $\mathfrak{t}$ *into* $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term

$$\mathfrak{t}' := \mathrm{c}_2 \mathrm{v}_3 \lfloor \mathrm{c}_3 \lfloor \mathrm{c}_1 \mathrm{v}_2 \rfloor \lfloor \mathrm{c}_2 \mathrm{v}_3 \lfloor \mathrm{c}_1 \mathrm{c}_0 \rfloor \rfloor \lfloor \mathrm{c}_1 \mathrm{v}_2 \rfloor \rfloor.$$

- Let $\mathfrak{t} := \mathrm{c}_3 \mathrm{v}_1 \lfloor \mathrm{c}_2 \mathrm{v}_2 \mathrm{v}_3 \rfloor \mathrm{v}_1$. Since $\mathfrak{t}$ is a prefix of $\mathfrak{t}' \cdot 2$, $\mathfrak{t}$ is a factor of $\mathfrak{t}'$ and 2 is an occurrence of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Let $\mathfrak{t} := \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_2$. Since $\mathfrak{t}$ is a prefix of $\mathfrak{t}'$ and of $\mathfrak{t}' \cdot 22$, ϵ and 22 are two occurrences of $\mathfrak{t}$ into $\mathfrak{t}'$.
- Let $\mathfrak{t} := \mathrm{c}_2 \mathrm{v}_1 \mathrm{v}_1$. There is no position w within $\mathfrak{t}'$ such that $\mathfrak{t}$ is a prefix of $\mathfrak{t}' \cdot w$. Therefore, $\mathfrak{t}$ is not a factor of $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

A *holed* $\mathcal{S}, \mathsf{V}$-term is an $\mathcal{S}, \mathsf{V} \sqcup \{\square\}$-term $\mathfrak{t}$ such that $\ell_{\square} \cdot \mathfrak{t} = 1$ and $\square$ is a variable which does not belong to V. The *hole position* of $\mathfrak{t}$ is the position within $\mathfrak{t}$ of its unique leaf decorated by $\square$.

For any holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, and any $\mathcal{S}, \mathsf{V}$-substitution σ, let

$$\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma := \mathfrak{s} \curvearrowleft_{\square} \lfloor \bar{\sigma} \cdot \mathfrak{t} \rfloor.$$

Proposition [Factors of $\mathcal{S}, \mathsf{V}$-terms]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms, and w be a position within $\mathfrak{t}'$. The two following assertions are equivalent:

1. w is an occurrence of $\mathfrak{t}$ into $\mathfrak{t}'$;
2. there exists a holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$ having w as hole position and an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\mathfrak{t}' := \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma$.

Example

By considering the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}$-terms $\mathfrak{t}'$ and $\mathfrak{t}$ of the previous first example, we have

$$\mathfrak{t}' = \blacktriangle \cdot \lfloor \mathsf{c}_2 \mathsf{v}_3 \square \rfloor \cdot \mathfrak{t} \cdot [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_2, \mathsf{v}_3), (\mathsf{v}_3, \mathsf{c}_1 \mathsf{c}_0)\}].$$

Exercise ●○○○○

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}' := \mathsf{c}_3 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_2 \underline{\mathsf{c}_2 \underline{\mathsf{c}_1 \mathsf{v}_3} \mathsf{c}_0}$. Give examples of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ are factors of $\mathfrak{t}'$ of constant lengths n for all $n \in [\![5]\!]$.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. For any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, show that $\mathfrak{t} \preccurlyeq_{\mathrm{p}} \mathfrak{t}'$ implies $\mathfrak{t} \preccurlyeq_{\mathrm{f}} \mathfrak{t}'$. Show that the converse of this property is false.

Exercise ●○○○○

Show that the set of factors of an $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term is infinite.

Exercise ●●●○○

For any signature $\mathcal{S}$ and any labeled $\mathcal{S}$-term $\mathfrak{t}'$, prove that the set of planar labeled $\mathcal{S}$-signature terms $\mathfrak{t}$ such that $\mathfrak{t}$ is a factor of $\mathfrak{t}'$ is finite.

Proposition [Rewrite relation of a TRS]

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS. The rewrite relation $\Rightarrow$ of $\mathcal{T}$ is characterized by the fact that

$$\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma \quad \Rightarrow \quad \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t}' \cdot \sigma$$

where

1. $\mathfrak{s}$ is a holed $\mathcal{S}, \mathsf{V}$-term;
2. $\mathfrak{t}$ and $\mathfrak{t}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t} \to \mathfrak{t}'$;
3. σ is an $\mathcal{S}, \mathsf{V}$-substitution.

Let us consider the notations used in the statement of the previous proposition.

The **hole position** w of $\mathfrak{s}$ is the *rewrite position* of the **one-step rewrite** from $\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma$ to $\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t}' \cdot \sigma$ in $\mathcal{T}$. Note that this position w is also a position within $\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma$ and $\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t}' \cdot \sigma$.

This property is written as $\blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma \Rightarrow_w \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t}' \cdot \sigma$.

The binary relation $\Rightarrow_\epsilon$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ is the *one-step root rewrite relation* of $\mathcal{T}$.

Example

In NatAdd, we have

$$a\lfloor az\lfloor s\lfloor sz\rfloor\rfloor\rfloor\lfloor s1\rfloor \Rightarrow a\lfloor s\lfloor az\lfloor sz\rfloor\rfloor\rfloor\lfloor s1\rfloor$$

because

$$a\lfloor az\lfloor s\lfloor sz\rfloor\rfloor\rfloor\lfloor s1\rfloor = \blacktriangle \cdot a\square\lfloor s1\rfloor \cdot \lfloor a1\lfloor s2\rfloor\rfloor \cdot [z, sz] \quad\Rightarrow\quad \blacktriangle \cdot a\square\lfloor s1\rfloor \cdot \lfloor s\lfloor a12\rfloor\rfloor \cdot [z, sz] = a\lfloor s\lfloor az\lfloor sz\rfloor\rfloor\rfloor\lfloor s1\rfloor.$$

Example

In Eq, we have

$$e\lfloor et3\rfloor\lfloor et3\rfloor \Rightarrow t$$

because

$$e\lfloor et3\rfloor\lfloor et3\rfloor = \blacktriangle \cdot \square \cdot \lfloor e11\rfloor \cdot [et3] \quad\Rightarrow\quad \blacktriangle \cdot \square \cdot t \cdot [et3] = t.$$

Let TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS.

Given $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$, there **can exist different positions** w_1 and w_2 within $\mathfrak{t}$ such that $w_1 \neq w_2$, $\mathfrak{t} \Rightarrow_{w_1} \mathfrak{t}'$, and $\mathfrak{t} \Rightarrow_{w_2} \mathfrak{t}'$.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ such that $\mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3 \to \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2$ and $\mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3 \to \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3$.
We have $\mathsf{c}_2 \underline{\mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3} \mathsf{v}_3 \Rightarrow_{\epsilon} \mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3$ and $\mathsf{c}_2 \underline{\mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3} \mathsf{v}_3 \Rightarrow_{1} \mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3$.

In order to **keep track of these multiplicities**, let $\mathrm{f}_{\Rightarrow} \cdot \mathfrak{t}$ be the $\mathbb{K}, \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$-series, where $\mathbb{K}$ is any field, defined by

$$\mathrm{f}_{\Rightarrow} \cdot \mathfrak{t} := \sum_{w \in \mathrm{P} \cdot \mathfrak{t}} \; \sum_{\mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}} [\mathfrak{t} \Rightarrow_w \mathfrak{t}'] \, \mathfrak{t}'.$$

Example

By considering the TRS $\mathcal{T}$ of the previous example, we have

$$\mathrm{f}_{\Rightarrow} \cdot \mathsf{c}_2 \underline{\mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3} \mathsf{v}_3 = 2 \, \mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3 + \mathsf{c}_3 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3 \mathsf{v}_3 + \mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \mathsf{v}_3.$$

Proposition [Compatibilities between rewrite relations and substitutions]

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS, $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms, and σ and σ' be two $\mathcal{S}, \mathsf{V}$-substitutions.

1. If $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\overline{\sigma} \cdot \mathfrak{t} \Rightarrow \overline{\sigma} \cdot \mathfrak{t}'$.
2. For any holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\mathfrak{s} \curvearrowleft_{\square} \mathfrak{t} \Rightarrow \mathfrak{s} \curvearrowleft_{\square} \mathfrak{t}'$.
3. For any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$ and any $v \in \mathsf{V}$, if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then $\mathfrak{s} \curvearrowleft_{v} \mathfrak{t} \Rightarrow^{*} \mathfrak{s} \curvearrowleft_{v} \mathfrak{t}'$.
4. If, for all $v \in \mathrm{Vars} \cdot \mathfrak{t}$, $\sigma \cdot v \Rightarrow \sigma' \cdot v$, then $\overline{\sigma} \cdot \mathfrak{t} \Rightarrow^{*} \overline{\sigma'} \cdot \mathfrak{t}$.

Exercise ●○○○○

Define a TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ and provide a nontrivial example for each property 1., 2., 3., and 4. of the previous proposition.

Exercise ●○○○○

Build an example of a ‘‘TRS’’ $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ such that $\mathcal{T}$ does not satisfy 1. and such that $\mathcal{T}$ does not satisfy Property 1. of the previous proposition.

7. Termination

7.1. Normal forms and pattern avoidance

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. When $\mathfrak{t}$ is **not a factor** of $\mathfrak{t}'$, $\mathfrak{t}'$ *avoids* $\mathfrak{t}$.

By extension, for any set X of $\mathcal{S}, \mathsf{V}$-terms, an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}'$ *avoids* X if for any $\mathfrak{t} \in X$, $\mathfrak{t}'$ avoids $\mathfrak{t}$.

Examples

On the signature $\mathcal{S}_{\mathbb{N}^2}$ and the set of variables $\mathsf{V}_{\mathbb{N}}$, let $X := \{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1, \mathsf{c}_2 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \mathsf{v}_3, \mathsf{c}_3 \mathsf{v}_1 \underline{\mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3} \mathsf{v}_4\}$.

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

- ☐ $\mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \underline{\mathsf{c}_2 \mathsf{v}_4 \mathsf{v}_5}$ avoids X;
- ☐ $\mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3}$ does not avoid X;
- ☐ $\mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_2}$ avoids X;
- ☐ $\mathsf{c}_2 \mathsf{v}_1 \underline{\underline{\mathsf{c}_2 \underline{\mathsf{c}_3 \mathsf{v}_1 \underline{\mathsf{v}_2 \mathsf{v}_3 \mathsf{v}_1} \mathsf{v}_2}} \underline{\mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_1 \underline{\mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_4}}}$ avoids X.

Exercise ●○○○○

On the signature $\mathcal{S}_{\mathbb{N}^2}$ and the set of variables $\mathsf{V}_{\mathbb{N}}$, let $X := \{\mathsf{c}_n \mathsf{v}_1 \dots \mathsf{v}_1 : n \in \mathbb{N}\}$. Give an example of a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of constant length 7 avoiding the set X and an example of a $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of constant length 5 which does not avoid X.

Given a TRS $\mathcal{T}$, let $\mathrm{L}{\cdot}\mathcal{T}$ be the set of left-hand sides of rewrite rules of $\mathcal{T}$.

Theorem [Normal forms and avoidance]

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. The set of normal forms of $\mathcal{T}$ is the set of $\mathcal{S}, \mathrm{V}$-terms avoiding $\mathrm{L}{\cdot}\mathcal{T}$.

Proof. Let $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term. The property for $\mathfrak{t}$ to be a normal form is equivalent to the fact that there is no $\mathcal{S}, \mathrm{V}$-term $\mathfrak{t}'$ such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$. By Propositions [Factors of $\mathcal{S}, \mathrm{V}$-terms] and [Rewrite relation of a TRS], this is equivalent to the fact that $\mathfrak{t}$ has no factor $\mathfrak{s}$ such that $\mathfrak{s}$ is the left-hand side of a rewrite rule of $\mathcal{T}$. This property is finally equivalent to the fact that $\mathfrak{t}$ avoids $\mathrm{L}{\cdot}\mathcal{T}$.

Examples

- The normal forms of the TRS Assoc avoid $\{m \llcorner \underline{m12} \lrcorner 3\}$.
- The normal forms of the TRS Assoc_2 avoid $\{m \llcorner \underline{m12} \lrcorner 3,\ m'1 \llcorner \underline{m'23} \lrcorner\}$.
- The normal forms of the TRS Eq avoid $\{e11\}$.
- the normal forms of the TRS NatAdd avoid $\{a1z,\ a1 \llcorner \underline{s2} \lrcorner\}$.

Let us consider the following **combinatorial problem** consisting of the following steps:

1. consider a signature $\mathcal{S}$, a set of variables V and a set X of $\mathcal{S}, \mathsf{V}$-terms;
2. define Y as the set of $\mathcal{S}, \mathsf{V}$-terms avoiding X;
3. let Y' be the subset of Y of $\mathcal{S}, \mathsf{V}$-terms satisfying some property (like being ground, linear, or having variables in a finite subset V' of V);
4. consider a rank function rk (like ℓ, ℓ_{cns}, or ℓ_{var}) so that $\mathcal{G} := (Y', \mathrm{rk})$ is combinatorial;
5. describe the integer sequence of $\mathcal{G}$.

Examples

Let us consider the enumeration of graded sets $\mathcal{G}$ of planar labeled $\mathcal{S}_{\mathbb{N}^2}$-terms where rank function is ℓ_{var} and avoiding some sets X of labeled $\mathcal{S}_{\mathbb{N}^2}$-terms:

- if $X = \left\{\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \ \mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3, \ \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \ \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3\right\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 2, 4, 8, 16, 32, 64, 128 (**powers of 2**, **A000079**);
- if $X = \{\mathsf{c}_2 \lfloor \mathsf{c}_2 12 \rfloor 3, \ \mathsf{c}_2 \lfloor \mathsf{c}_3 123 \rfloor 4, \ \mathsf{c}_3 \lfloor \mathsf{c}_2 12 \rfloor 34, \ \mathsf{c}_3 \lfloor \mathsf{c}_3 123 \rfloor 45\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 1, 2, 4, 9, 21, 51, 127 (**Motzkin numbers**, **A001006**);
- if $X = \left\{\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \ \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3, \ \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \ \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,1} 2 \lfloor \mathsf{c}_{2,1} 34 \rfloor \rfloor\right\}$, then the integer sequence of $\mathcal{G}$ starts by 1, 2, 5, 13, 35, 96, 267, 750 (**directed animals**, **A005773**).

There are some known results for avoidance of sets of

- **binary trees** [E. S. Rowland, Pattern avoidance in binary trees, 2010];
- **ternary trees** [N. Gabriel, K. Peske, L. Pudwell, S. Tay, Pattern avoidance in ternary trees, 2012];
- **planar labeled $\mathcal{S}$-terms of constant length 2** [S. F. Parker, The combinatorics of functional composition and inversion, 1993], [J.-L. Loday, Inversion of integral series enumerating planar trees, 2005];
- **planar labeled $\mathcal{S}$-terms** [A. Khoroshkin and D. Piontkovski, On generating series of finitely presented operads, 2015], [S. Giraudo, Tree series and pattern avoidance in syntax trees, 2020].

Term series and their **grafting operations** are powerful tools to describe the characteristic series of such $\mathcal{S}, \mathsf{V}$-terms.

Exercise ○○○○○

Given a signature $\mathcal{S}$, a set of variables V, and a set of $\mathcal{S}, \mathsf{V}$-terms X, provide a systems of equations for the characteristic series of the $\mathcal{S}, \mathsf{V}$-terms avoiding X.

The main difficulty of this **research question** comes from the avoidance of $\mathcal{S}, \mathsf{V}$-terms which are not linear.

Exercise ●●●○○

Let N be the set of variables $\mathbb{N} \setminus \{0\}$ and, for any $m \in \mathbb{N}$, $\mathcal{S}_m$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_{2,i} : i \in [\![m]\!]\}$.

For any $m \in \mathbb{N}$, let the TRS $\mathsf{FCat}_m := (\mathcal{S}_m, N, \to_m)$ such that $\to_m$ satisfies

$$\mathsf{c}_{2,i+j}\lfloor \mathsf{c}_i 12 \rfloor 3 \to_m \mathsf{c}_{2,i} 1 \lfloor \mathsf{c}_{2,j} 23 \rfloor$$

for any $i, j \in [\![m]\!]$ with $i + j \in [\![m]\!]$.

Describe a system of equations in $\mathbb{K}\langle\langle \mathfrak{T} \cdot \mathcal{S}_m \cdot N \rangle\rangle$ for the planar normal forms of FCat_m, $m \in \mathbb{N}$.

Exercise ●●●○○

Let N be the set of variables $\mathbb{N} \setminus \{0\}$ and $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_{2,0}, \mathsf{c}_{2,1}, \mathsf{c}_{2,2}\}$.

Let the TRS $\mathsf{Schr} := (\mathcal{S}, N, \to)$ such that $\to$ satisfies

$$\mathsf{c}_{2,0}\lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,1}\lfloor \mathsf{c}_{2,2} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,1} 23 \rfloor, \quad \mathsf{c}_{2,0}\lfloor \mathsf{c}_{2,1} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,2} 23 \rfloor, \quad \mathsf{c}_{2,1}\lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,1} 23 \rfloor,$$

$$\mathsf{c}_{2,0}\lfloor \mathsf{c}_{2,2} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,1}\lfloor \mathsf{c}_{2,1} 12 \rfloor 3 \to \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,2}\lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,2} 23 \rfloor.$$

Describe a system of equations in $\mathbb{K}\langle\langle \mathfrak{T} \cdot \mathcal{S} \cdot N \rangle\rangle$ for the planar normal forms of Schr.

7.2. Reduction relations

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

A binary relation $\rightsquigarrow$ is a *termination witness* of $\mathcal{A}$ if the ARS $\mathcal{A}' := (X, \rightsquigarrow)$ is **terminating** and $\mathcal{A}$ is a **sub-ARS** of $\mathcal{A}'$.

Proposition [Termination witnesses]

An ARS $\mathcal{A}$ is terminating iff $\mathcal{A}$ admits a termination witness.

Proof. Let us denote by $\Rightarrow$ the rewrite relation of $\mathcal{A}$.

Assume that $\mathcal{A}$ is terminating. In this case, $\Rightarrow$ is trivially a termination witness of $\mathcal{A}$.

Conversely, assume that $\mathcal{A}$ admits a termination witness $\rightsquigarrow$. Assume that $(u_i)_{i \in \mathbb{N}}$ is an infinite rewrite sequence in $\mathcal{A}$. For any $i \in \mathbb{N}$, $u_i \Rightarrow u_{i+1}$. Since, by hypothesis, $\mathcal{A}$ is a sub-ARS of $\mathcal{A}' := (X, \rightsquigarrow)$, we have $\Rightarrow \subseteq \rightsquigarrow$ so that, for any $i \in \mathbb{N}$, $u_i \rightsquigarrow u_{i+1}$. This shows that $(u_i)_{i \in \mathbb{N}}$ is also an infinite rewrite sequence in $\mathcal{A}'$. This contradicts our hypothesis that $\mathcal{A}'$ is terminating, showing that such an infinite rewrite sequence $(u_i)_{i \in \mathbb{N}}$ in $\mathcal{A}$ does not exist. Therefore, $\mathcal{A}$ is terminating.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a binary relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

The binary relation $\rightsquigarrow$ is *compatible from factors* if $\rightsquigarrow$ satisfies the two following properties:

1. for any $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $i \in [n]$, $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$,
$$\mathfrak{t}_i \rightsquigarrow \mathfrak{t}'_i \quad \text{implies} \quad c\,\mathfrak{t}_1 \dots \mathfrak{t}_i \dots \mathfrak{t}_n \rightsquigarrow c\,\mathfrak{t}_1 \dots \mathfrak{t}'_i \dots \mathfrak{t}_n;$$
2. for any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ and any $\mathcal{S}, \mathsf{V}$-substitution σ,
$$\mathfrak{t} \rightsquigarrow \mathfrak{t}' \quad \text{implies} \quad \overline{\sigma}{\cdot}\mathfrak{t} \rightsquigarrow \overline{\sigma}{\cdot}\mathfrak{t}'.$$

Proposition [Compatibility from factors]

Let $\mathcal{S}$ be a signature and V be a set of variables. A binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ is compatible from factors iff for any holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and any $\mathcal{S}, \mathsf{V}$-substitution σ,
$$\mathfrak{t} \rightsquigarrow \mathfrak{t}' \quad \text{implies} \quad \blacktriangle{\cdot}\mathfrak{s}{\cdot}\mathfrak{t}{\cdot}\sigma \rightsquigarrow \blacktriangle{\cdot}\mathfrak{s}{\cdot}\mathfrak{t}'{\cdot}\sigma.$$

Definition

Let $\mathcal{S}$ be a signature and V be a set of variables. A binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ is a *reduction relation* on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ if the ARS $(\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}, \rightsquigarrow)$ is terminating and $\rightsquigarrow$ is compatible from factors.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *length relation* is the binary relation $\rightsquigarrow_\ell$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathrm{V}$ defined, for any $\mathcal{S}, \mathrm{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, by $\mathfrak{t} \rightsquigarrow_\ell \mathfrak{t}'$ if $\ell{\cdot}\mathfrak{t} > \ell{\cdot}\mathfrak{t}'$ and, for any $v \in \mathrm{V}$, $\ell_v{\cdot}\mathfrak{t} \geqslant \ell_v{\cdot}\mathfrak{t}'$.

Examples

On $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms, we have

- $\mathfrak{t} := \mathsf{c}_2 \lfloor \mathsf{c}_4 \mathsf{v}_2 \mathsf{v}_1 \mathsf{v}_1 \mathsf{v}_1 \rfloor \mathsf{v}_3 \rightsquigarrow_\ell \mathsf{c}_3 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1 \rfloor \mathsf{c}_0 \mathsf{c}_0 := \mathfrak{t}'$ because $\ell{\cdot}\mathfrak{t} = 7 > 6 = \ell{\cdot}\mathfrak{t}'$, $\ell_{\mathsf{v}_1}{\cdot}\mathfrak{t} = 3 \geqslant 2 = \ell_{\mathsf{v}_1}{\cdot}\mathfrak{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}{\cdot}\mathfrak{t}'$, and $\ell_{\mathsf{v}_3}{\cdot}\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_3}{\cdot}\mathfrak{t}'$.
- By setting $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_2$ and $\mathfrak{t}' := \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3$, we have $\mathfrak{t} \not\rightsquigarrow_\ell \mathfrak{t}'$ and $\mathfrak{t}' \not\rightsquigarrow_\ell \mathfrak{t}$.

Proposition [Length reduction relation]

For any signature $\mathcal{S}$ and any set of variables V, $\rightsquigarrow_\ell$ is a reduction relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathrm{V}$.

Exercise ●●●○○

Prove the previous proposition.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS.

A binary relation $\rightsquigarrow$ on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}$ is $\mathcal{T}$*-compatible* if $\rightarrow \subseteq \rightsquigarrow$.

Theorem [Compatible reduction relations and termination]

A TRS $\mathcal{T}$ is terminating iff there exists a $\mathcal{T}$-compatible reduction relation.

Proof. Let $\mathcal{S}$ be the underlying signature of $\mathcal{T}$, V the underlying set of variables of $\mathcal{T}$, and $\rightarrow$ be the elementary rewrite relation of $\mathcal{T}$.

Assume that $\mathcal{T}$ is terminating. By Propositions [Rewrite relation of a TRS] and [Compatibility from factors], $\Rightarrow$ is compatible from factors. Moreover, $\Rightarrow$ is trivially $\mathcal{T}$-compatible. Since by hypothesis, the ARS $(\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}, \Rightarrow)$ is terminating, $\Rightarrow$ is a $\mathcal{T}$-compatible reduction relation.

Assume that $\rightsquigarrow$ is a $\mathcal{T}$-compatible reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}$. Since $\rightsquigarrow$ is a reduction relation, the ARS $\mathcal{A} := (\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}, \rightsquigarrow)$ is terminating. Moreover, assume that $\mathfrak{r}$ and $\mathfrak{r}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{r} \Rightarrow \mathfrak{r}'$. By Proposition [Rewrite relation of a TRS], there is a holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\mathfrak{t} \rightarrow \mathfrak{t}'$, $\mathfrak{r} = \blacktriangle\cdot\mathfrak{s}\cdot\mathfrak{t}\cdot\sigma$, and $\mathfrak{r}' = \blacktriangle\cdot\mathfrak{s}\cdot\mathfrak{t}'\cdot\sigma$. From this, and since $\rightsquigarrow$ is $\mathcal{T}$-compatible, we have $\mathfrak{t} \rightsquigarrow \mathfrak{t}'$. Now, since $\rightsquigarrow$ is compatible from factors, by Proposition [Compatibility from factors], the previous properties imply $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$. This shows that the ARS of $\mathcal{T}$ is a sub-ARS of $\mathcal{A}$. By Proposition [Termination witnesses], the desired property holds.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ is terminating, the *reduction relation method* consists in

1. constructing a binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$;
2. showing that the ARS $(\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}, \rightsquigarrow)$ is terminating;
3. showing that $\rightsquigarrow$ is compatible from factors;
4. showing that $\rightsquigarrow$ is $\mathcal{T}$-compatible.

By Theorem [Compatible reduction relations and termination], we obtain the desired property.

In practice, it is difficult to guess such a binary relation $\rightsquigarrow$ which will satisfy the three required properties.

Example

Let us show that the TRS Assoc is terminating by using the reduction relation method.

Let $\mathcal{S}$ (resp. V) be the underlying signature (resp. set of variables) of Assoc.

1. Let for any $v \in \mathrm{V}$ the function lc defined for any $\mathcal{S}, \mathrm{V}$-term t by

$$\mathrm{lc}_v \cdot \mathrm{t} := \sum_{w \in \mathrm{P} \cdot \mathrm{t}} [\mathrm{w} \cdot \underline{\mathrm{t} \cdot w} \cdot 1 = v] \, 2^{\ell_1 \cdot w}.$$

   For instance, $\mathrm{lc}_1 \cdot \underline{m \underline{m \underline{m12} \underline{m \underline{m13} 1}} \underline{m22}} = 8 + 8 + 1 = 17$.

   Let $\rightsquigarrow$ be the binary relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$ such that, for any $\mathcal{S}, \mathrm{V}$-terms t and t′, $\mathrm{t} \rightsquigarrow \mathrm{t}'$ if for all $v \in \mathrm{V}$, $\mathrm{lc}_v \cdot \mathrm{t} \geqslant \mathrm{lc}_v \cdot \mathrm{t}'$ and there exists $v \in \mathrm{V}$ such that $\mathrm{lc}_v \cdot \mathrm{t} > \mathrm{lc}_v \cdot \mathrm{t}'$.
2. Let the function $\theta := \mathrm{t} \mapsto \sum_{v \in \mathrm{V}} \mathrm{lc}_v \cdot \mathrm{t}$. For any $\mathcal{S}, \mathrm{V}$-terms t and t′ such that $\mathrm{t} \rightsquigarrow \mathrm{t}'$, $\theta \cdot \mathrm{t} > \theta \cdot \mathrm{t}'$. Since the codomain of θ is $\mathbb{N}$, the ARS $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}, \rightsquigarrow)$ is terminating.
3. The compatibility from factors is straightforward (but technical): Properties 1. and 2. can be proven directly.
4. Let $\mathrm{t} := m \underline{m12} 3$ and $\mathrm{t}' := m1 \underline{m23}$. Since $(\mathrm{lc}_1 \cdot \mathrm{t}, \mathrm{lc}_2 \cdot \mathrm{t}, \mathrm{lc}_3 \cdot \mathrm{t}) = (4, 2, 1)$ and $(\mathrm{lc}_1 \cdot \mathrm{t}', \mathrm{lc}_2 \cdot \mathrm{t}', \mathrm{lc}_3 \cdot \mathrm{t}') = (2, 2, 1)$, $\mathrm{t} \rightsquigarrow \mathrm{t}'$. Therefore, $\rightsquigarrow$ is Assoc-compatible.

7.3. Semantic method

Definition

An $\mathcal{S}$-*algebra* is a triple $(X, \mathcal{S}, \mathrm{op})$ where

- $\mathcal{S}$ is a signature, called the *underlying signature*;
- X is a nonempty set, called the *underlying set*;
- op is a function such that for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $\mathrm{op} \cdot c$ is an n-operation on X.

Examples

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ such that $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

- By setting $\mathrm{op} \cdot z := 0$, $\mathrm{op} \cdot s := n \mapsto 1 + n$, and $\mathrm{op} \cdot a := n_1 \mapsto n_2 \mapsto n_1 + n_2$, the triple $(\mathbb{N}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.
- By setting $\mathrm{op} \cdot z := \emptyset$, $\mathrm{op} \cdot s := S \mapsto \mathbb{Z} \setminus S$, and $\mathrm{op} \cdot a := S_1 \mapsto S_2 \mapsto S_1 \cap S_2$, the triple $(\mathcal{P} \cdot \mathbb{Z}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.
- By setting $\mathrm{op} \cdot z := 1$, $\mathrm{op} \cdot s := b \mapsto 1 - b$, and $\mathrm{op} \cdot a := b_1 \mapsto b_2 \mapsto b_1 \times b_2$, the triple $(\{0, 1\}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.

Let V be a set of variables and X be a nonempty set.

A V, X*-assignment* is a function $\alpha : \mathsf{V} \to X$.

Given a signature $\mathcal{S}$, an $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, and a V, X-assignment α, the $\mathcal{A}, \alpha$*-evaluation* $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}$ of an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ is defined recursively by

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} := \begin{cases} \alpha \cdot v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ \mathrm{op} \cdot c \cdot \underbrace{\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_1} \cdot \cdots \cdot \underbrace{\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_n} & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, c \in \mathcal{S} \cdot n, n \in \mathbb{N}, \mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}. \end{cases}$$

Example

Let us consider the first $\mathcal{S}$-algebra $\mathcal{A}$ of the previous example.
Let α be the $\mathsf{V}_{\mathbb{N}}, \mathbb{N}$-assignment α defined by $\alpha \cdot \mathsf{v}_1 := 3$, $\alpha \cdot \mathsf{v}_2 := 1$, and $\alpha \cdot \mathsf{v}_i := 0$, $i \geqslant 3$. We have

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \underbrace{s\, \underbrace{a\, \underbrace{s\, \mathsf{v}_1}\, \underbrace{a\, \underbrace{s\, \underbrace{s\, z}}\, \mathsf{v}_2}}} = 1 + \underbrace{1 + 3} + \underbrace{1 + \underbrace{1 + 0} + 1} = 8.$$

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X.

The *relation induced by* $\mathcal{R}$ is the binary relation $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ such that, for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{t}'$ holds if $\mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\mathfrak{t} \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\mathfrak{t}'$ for all V, X-assignments α.

Examples

Let us consider the $\mathcal{S}$-algebra $\mathcal{A}$ of the previous example.

Let $>$ be the usual ‘‘greater-than’’ relation on $\mathbb{N}$.

- We have

$$a \mathsf{v}_1 \llcorner s \mathsf{v}_2 \lrcorner \quad \rightsquigarrow_{\mathcal{A},>} \quad a \mathsf{v}_1 \mathsf{v}_2.$$

  Indeed, for any $\mathsf{V}_{\mathbb{N}}, \mathbb{N}$ assignment α, we have

$$\mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\llcorner a \mathsf{v}_1 \llcorner s \mathsf{v}_2 \lrcorner \lrcorner = \llcorner \alpha{\cdot}\mathsf{v}_1 \lrcorner + \llcorner \alpha{\cdot}\mathsf{v}_2 \lrcorner + 1 > \llcorner \alpha{\cdot}\mathsf{v}_1 \lrcorner + \llcorner \alpha{\cdot}\mathsf{v}_2 \lrcorner = \mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\llcorner a \mathsf{v}_1 \mathsf{v}_2 \lrcorner.$$

- By setting $\mathfrak{t} := a \mathsf{v}_1 \mathsf{v}_2$ and $\mathfrak{t}' := s \mathsf{v}_1$, we have neither $\mathfrak{t} \rightsquigarrow_{\mathcal{A},>} \mathfrak{t}'$ nor $\mathfrak{t}' \rightsquigarrow_{\mathcal{A},>} \mathfrak{t}$.

Let $\mathcal{S}$ be a signature, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X.

The $\mathcal{S}$-algebra $\mathcal{A}$ is $\mathcal{R}$*-monotone* if for any $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $i \in [n]$, and $x_1, \dots, x_i, x'_i, \dots, x_n \in X$,

$$x_i \,\mathcal{R}\, x'_i \quad \text{implies} \quad \mathrm{op}{\cdot}c{\cdot}x_1{\cdot} \cdots {\cdot}x_i{\cdot} \cdots {\cdot}x_n \;\mathcal{R}\; \mathrm{op}{\cdot}c{\cdot}x_1{\cdot} \cdots {\cdot}x'_i{\cdot} \cdots {\cdot}x_n.$$

Example

Let us consider the $\mathcal{S}$-algebra $\mathcal{A}$ and the binary relation $>$ of the previous example. We immediately have that $\mathcal{A}$ is $>$-monotone.

Theorem [Reduction relations from $\mathcal{S}$-algebras]

Let $\mathcal{S}$ be a signature, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X. If the ARS $(X, \mathcal{R})$ is terminating and $\mathcal{A}$ is $\mathcal{R}$-monotone, then $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ is a reduction relation.

This result is due to [Z. Manna, S. Ness, On the termination of Markov algorithms, 1970].

Proof (of Theorem [Reduction relations from $\mathcal{S}$-algebras]). Since X is nonempty, pick $x \in X$ and let α be the V, X-assignment defined by $\alpha \cdot v := x$ for any $v \in \mathsf{V}$. Let $(u_i)_{i \in \mathbb{N}}$ be an infinite rewrite sequence in $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow_{\mathcal{A}, \mathcal{R}})$. Hence, for any $i \in \mathbb{N}$, $u_i \rightsquigarrow_{\mathcal{A}, \mathcal{R}} u_{i+1}$, so that, by definition of $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$, for any $i \in \mathbb{N}$, $\mathrm{ev}_{\mathcal{A}, \alpha} \cdot u_i \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A}, \alpha} \cdot u_{i+1}$. Therefore, $(\mathrm{ev}_{\mathcal{A}, \alpha} \cdot u_i)_{i \in \mathbb{N}}$ is an infinite rewrite sequence in $(X, \mathcal{R})$. Since $(X, \mathcal{R})$ is terminating, such an infinite rewrite sequence $(u_i)_{i \in \mathbb{N}}$ cannot exists in $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow_{\mathcal{A}, \mathcal{R}})$. Hence, this ARS is terminating.

Let $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathfrak{t}_i \rightsquigarrow_{\mathcal{A}, \mathcal{R}} \mathfrak{t}'_i$. By definition of $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$, for any V, X-assignment α, $\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_i \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}'_i$. Since $\mathcal{A}$ is $\mathcal{R}$-monotone, by setting $\mathfrak{s} := c \, \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_i \cdot \cdots \cdot \mathfrak{t}_n$ and $\mathfrak{s}' := c \, \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}'_i \cdot \cdots \cdot \mathfrak{t}_n$, we have $\mathfrak{s} \rightsquigarrow_{\mathcal{A}, \mathcal{R}} \mathfrak{s}'$ because

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{s} = \mathrm{op} \cdot c \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_1 \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_i \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_n \rfloor \; \mathcal{R} \; \mathrm{op} \cdot c \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_1 \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}'_i \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{t}_n \rfloor = \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{s}'.$$

Given a V, X-assignment α and an $\mathcal{S}, \mathsf{V}$-substitution σ, let $\alpha \circ \sigma$ be the V, X-assignment defined, for any $v \in \mathsf{V}$, by $\lfloor \alpha \circ \sigma \rfloor \cdot v := \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \lfloor \sigma \cdot v \rfloor$. It follows, by structural induction on an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ that $\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \lfloor \overline{\sigma} \cdot \mathfrak{t} \rfloor = \mathrm{ev}_{\mathcal{A}, \alpha \circ \sigma} \cdot \mathfrak{t}$. Now, let $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathfrak{t} \rightsquigarrow_{\mathcal{A}, \mathcal{R}} \mathfrak{t}'$. From this last assumption, for any $\mathcal{S}, \mathsf{V}$-substitution σ, by setting $\mathfrak{s} := \overline{\sigma} \cdot \mathfrak{t}$ and $\mathfrak{s}' := \overline{\sigma} \cdot \mathfrak{t}'$, we have $\mathfrak{s} \rightsquigarrow_{\mathcal{A}, \mathcal{R}} \mathfrak{s}'$ because

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{s} = \mathrm{ev}_{\mathcal{A}, \alpha \circ \sigma} \cdot \mathfrak{t} \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A}, \alpha \circ \sigma} \cdot \mathfrak{t}' = \mathrm{ev}_{\mathcal{A}, \alpha} \cdot \mathfrak{s}'.$$

This shows that $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$ is compatible from factors and implies the statement of the theorem.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ is terminating, the *semantic method* consists in

1. constructing an $\mathcal{S}$-algebra $(X, \mathcal{S}, \mathrm{op})$;
2. constructing a binary relation $\mathcal{R}$ on X;
3. showing that the ARS $(X, \mathcal{R})$ is terminating;
4. showing that $\mathcal{A}$ is $\mathcal{R}$-monotone;
5. showing that $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ is $\mathcal{T}$-compatible.

By Theorem [Reduction relations from $\mathcal{S}$-algebras], and Theorem [Compatible reduction relations and termination], we obtain the desired property.

In practice, it is difficult to guess such an $\mathcal{S}$-algebra $\mathcal{A}$ and such a binary relation $\mathcal{R}$ which will satisfy the three required properties.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}_{\mathbb{N}}, \to)$ be the TRS such that $\mathcal{S} := (\{z, s, f\}, \mathrm{ar})$ where $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot f = 1$, and $f \llcorner s \mathsf{v}_1 \lrcorner \to f \llcorner f \mathsf{v}_1 \lrcorner$. Let us show that $\mathcal{T}$ is terminating by using the semantic method.

1. Let the $\mathcal{S}$-algebra $\mathcal{A} := \left(\mathbb{N}^2, \mathcal{S}, \mathrm{op}\right)$ such that $\mathrm{op} \cdot z := (0, 0)$, $\mathrm{op} \cdot s := (n_1, n_2) \mapsto (n_1 + 1, n_2)$, and $\mathrm{op} \cdot f := (n_1, n_2) \mapsto (n_1, n_2 + 1)$.
2. Let the binary relation $\mathcal{R}$ on $\mathbb{N}^2$ defined by $(i_1, i_2) \, \mathcal{R} \, (i'_1, i'_2)$ if $i_1 > i'_1$, or $i_1 = i'_1$ and $i_2 > i'_2$.
3. It is immediate that $\left(\mathbb{N}^2, \mathcal{R}\right)$ is terminating.
4. It is immediate that $\mathcal{A}$ is $\mathcal{R}$-monotone.
5. For any $\mathsf{V}_{\mathbb{N}}, \mathbb{N}^2$-assignment α such that $\alpha \cdot \mathsf{v}_1 = (i_1, i_2)$, we have

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \llcorner f \llcorner s \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot s \cdot \llcorner \alpha \cdot \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot s \cdot (i_1, i_2) \lrcorner = \mathrm{op} \cdot f \cdot (i_1 + 1, i_2) = (i_1 + 1, i_2 + 1)$$

   and

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \llcorner f \llcorner f \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot f \cdot \llcorner \alpha \cdot \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot f \cdot (i_1, i_2) \lrcorner = \mathrm{op} \cdot f \cdot (i_1, i_2 + 1) = (i_1, i_2 + 2).$$

   Therefore, as $(i_1 + 1, i_2 + 1) \, \mathcal{R} \, (i_1, i_2 + 2)$, we have $f \llcorner s \mathsf{v}_1 \lrcorner \rightsquigarrow_{\mathcal{A}, \mathcal{R}} f \llcorner f \mathsf{v}_1 \lrcorner$. This shows that $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$ is $\mathcal{T}$-compatible.

7.4. Polynomial interpretations

Let X be the set of variables $\{\mathsf{x}_i : i \in \mathbb{N} \setminus \{0\}\}$ and X_n be the subset $\{\mathsf{x}_1, \dots, \mathsf{x}_n\}$ of X for any $n \in \mathbb{N}$.

Definition

Let $\mathcal{S}$ be a signature. An $\mathcal{S}$-*polynomial interpretation* is an $\mathcal{S}$-algebra $(N, \mathcal{S}, \mathrm{op})$ where

- N is a subset of $\mathbb{N} \setminus \{0\}$;
- for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, the function $\mathrm{op} \cdot c$ is functionally equivalent to a function $\mathsf{x}_1 \mapsto \cdots \mapsto \mathsf{x}_n \mapsto f_c$ such that f_c is a $\mathbb{K}, \mathrm{Mon} \cdot \mathsf{X}_n$-polynomial where coefficients belong to $\mathbb{N}$.

Example

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ where $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

The triple $(N, \mathcal{S}, \mathrm{op})$ where $\mathrm{op} \cdot z := 10$, $\mathrm{op} \cdot s := n_1 \mapsto n_1^2$, $\mathrm{op} \cdot a := n_1 \mapsto n_2 \mapsto 3n_1 + 2$, and $N := \{2n : n \in \mathbb{N} \setminus \{0\}\}$ is an $\mathcal{S}$-polynomial interpretation.

Indeed, $\mathrm{op} \cdot z$ is functionally equivalent to the constant function 10, $\mathrm{op} \cdot s$ is functionally equivalent to the function $\mathsf{x}_1 \mapsto \mathsf{x}_1^2$, and $\mathrm{op} \cdot a$ is functionally equivalent to the function $\mathsf{x}_1 \mapsto \mathsf{x}_2 \mapsto 3\mathsf{x}_1 + 2$.

We have $f_z = 10$, $f_s = \mathsf{x}_1^2$, and $f_a = 3\mathsf{x}_1 + 2$.

Let V be a set of variables and Z_{V} be the set of variables $\{\mathsf{z}_v : v \in \mathsf{V}\}$.

If f is a $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{X}_n$-polynomial, $n \in \mathbb{N}$, and $f'_1 \dots f'_n$ is a sequence of $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{Z}_{\mathsf{V}}$-polynomials, let $f(f'_1, \dots, f'_n)$ be the $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{Z}_{\mathsf{V}}$-polynomial obtained by substituting by f'_i each occurrence of a variable x_i in f, for all $i \in [n]$.

Let $\mathcal{S}$ be a signature and $(N, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-polynomial interpretation.

Given an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, the *polynomial* of $\mathfrak{t}$ is the $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{Z}_{\mathsf{V}}$-polynomial defined recursively by

$$\mathsf{Pol}{\cdot}\mathfrak{t} := \begin{cases} \mathsf{z}_v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ f_c(\mathsf{Pol}{\cdot}\mathfrak{t}_1, \dots, \mathsf{Pol}{\cdot}\mathfrak{t}_n) & \text{otherwise, where } \mathfrak{t} = c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c \in \mathcal{S}{\cdot}n, n \in \mathbb{N}, \mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}. \end{cases}$$

Example

Let $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ whose underlying set is $\{\mathsf{c}_0, \mathsf{c}_2, \mathsf{c}_3\}$ and the $\mathcal{S}$-polynomial interpretation $(N, \mathcal{S}, \mathrm{op})$ such that $N := \mathbb{N} \setminus \{0\}$, $\mathrm{op}{\cdot}\mathsf{c}_0 := 2$, $\mathrm{op}{\cdot}\mathsf{c}_2 := n_1 \mapsto n_2 \mapsto n_1^2 + n_2$, and $\mathrm{op}{\cdot}\mathsf{c}_3 := n_1 \mapsto n_2 \mapsto n_3 \mapsto n_1 n_2 + n_2 n_3$. We have $f_{\mathsf{c}_0} = 2$, $f_{\mathsf{c}_2} = \mathsf{x}_1^2 + \mathsf{x}_2$, and $f_{\mathsf{c}_3} = \mathsf{x}_1\mathsf{x}_2 + \mathsf{x}_2\mathsf{x}_3$.
Moreover,

$$\mathsf{Pol}{\cdot}\underline{\mathsf{c}_3\underline{\mathsf{c}_3\mathsf{v}_1\mathsf{v}_2\mathsf{v}_1}\underline{\mathsf{c}_2\mathsf{v}_3\mathsf{v}_5}\mathsf{c}_0} = f_{\mathsf{c}_3}(f_{\mathsf{c}_3}(\mathsf{z}_{\mathsf{v}_1}, \mathsf{z}_{\mathsf{v}_2}, \mathsf{z}_{\mathsf{v}_1}), f_{\mathsf{c}_2}(\mathsf{z}_{\mathsf{v}_3}, \mathsf{z}_{\mathsf{v}_5}), f_{\mathsf{c}_0})$$

$$= f_{\mathsf{c}_3}\left(2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}, \mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}, 2\right) = 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\left(\mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}\right) + 2\left(\mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}\right) = 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\mathsf{z}_{\mathsf{v}_3}^2 + 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\mathsf{z}_{\mathsf{v}_5} + 2\mathsf{z}_{\mathsf{v}_3}^2 + 2\mathsf{z}_{\mathsf{v}_5}.$$

Let $\mathcal{S}$ be a signature and $\mathcal{A} := (N, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-polynomial interpretation.

When, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, there is a monomial $\mathsf{x}_1^{e_1} \dots \mathsf{x}_i^{e_i} \dots \mathsf{x}_n^{e_n} \in \mathrm{Supp} \cdot f_c$ such that $e_i \geqslant 1$, $\mathcal{A}$ is *full*.

Example

The polynomial interpretation of the first example of this part is not full. Besides, by considering the signature $\mathcal{S}$ and the set N of this example, the triple $(N, \mathcal{S}, \mathrm{op})$ where $\mathrm{op} \cdot z := 1$, $\mathrm{op} \cdot s := n_1 \mapsto n_1$, and $\mathrm{op} \cdot a := n_1 \mapsto n_2 \mapsto n_1 + n_2^2 + 3 n_1 n_2$ is a full $\mathcal{S}$-polynomial interpretation.

Proposition [Monotonicity of full polynomial interpretations]

Let $\mathcal{S}$ be a signature and $\mathcal{A}$ be an $\mathcal{S}$-polynomial interpretation. If $\mathcal{A}$ is full, then $\mathcal{A}$ is $>$-monotone, where $>$ is the ‘‘greater-than’’ relation on integers restricted to the underlying set of $\mathcal{A}$.

Proof. Assume that $\mathcal{A} = (N, \mathcal{S}, \mathrm{op})$. Let $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, and $m_1, \dots, m_i, m_i', \dots, m_n \in N$ such that $m_i > m_i'$. Since $\mathcal{A}$ is full, there is a monomial $\mathsf{x}_1^{e_1} \dots \mathsf{x}_i^{e_i} \dots \mathsf{x}_n^{e_n}$ in the support of f_c with $e_i \geqslant 1$. Since for all $j \in [n] \setminus \{i\}$, $m_j \neq 0$, the inequality $m_i > m_i'$ implies $m_1^{e_1} \dots m_i^{e_i} \dots m_n^{e_n} > m_1^{e_1} \dots {m_i'}^{e_i} \dots m_n^{e_n}$. Therefore, and since each coefficient of f_c belongs to $\mathbb{N}$, $f_c(m_1, \dots, m_i, \dots, m_n) > f_c(m_1, \dots, m_i', \dots, m_n)$ so that $\mathrm{op} \cdot c \cdot m_1 \cdot \cdots \cdot m_i \cdot \cdots \cdot m_n > \mathrm{op} \cdot c \cdot m_1 \cdot \cdots \cdot m_i' \cdot \cdots \cdot m_n$. This shows the desired property.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ is terminating, the *polynomial interpretation method*, due to [D. Lankford, Canonical algebraic simplification in computational logic, 1975], consists in

1. constructing an $\mathcal{S}$-polynomial interpretation $\mathcal{A} := (N, \mathcal{S}, \mathrm{op})$;
2. showing that $\mathcal{A}$ is full;
3. showing that for any rewrite rule $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{T}$, the $\mathbb{K}, \mathrm{Mon}\cdot\mathsf{Z}_{\mathsf{V}}$-polynomial $\mathsf{Pol}\cdot\mathfrak{t} - \mathsf{Pol}\cdot\mathfrak{t}'$ is **positive** for any evaluation on N.

By Proposition [Monotonicity of full polynomial interpretations], Theorem [Reduction relation from $\mathcal{S}$-algebras], and Theorem [Compatible reduction relations and termination], we obtain the desired property.

At Step 1., we must be careful in the choice of N so that for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, all evaluations of f_c on N belong to N.

Step 3. amounts to show that for any rewrite rule $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{T}$ and any $\mathsf{V}, \mathbb{N}$-assignment α, $\mathrm{ev}_{\mathcal{A},\alpha}\cdot\mathfrak{t} > \mathrm{ev}_{\mathcal{A},\alpha}\cdot\mathfrak{t}'$.

Note that this decision problem is in general undecidable, as a consequence of the **Hilbert's Tenth Problem**.

Example

Let us prove the termination of the TRS Assoc_2 by the polynomial interpretation method.

Let $\mathcal{S}$ be the underlying signature of Assoc_2.

1. Let $\mathcal{A} := (N, \mathcal{S}, \mathrm{op})$ be the $\mathcal{S}$-polynomial interpretation such that $N := \mathbb{N} \setminus \{0\}$, $\mathrm{op} \cdot m := n_1 \mapsto n_2 \mapsto 2n_1 + n_2$, and $\mathrm{op} \cdot m' := n_1 \mapsto n_2 \mapsto n_1 + 2n_2$.
2. It is immediate that $\mathcal{A}$ is full.
3. We have

$$\mathsf{Pol} \cdot \lfloor m \lfloor m 1 2 \rfloor 3 \rfloor = 2(2\mathsf{z}_1 + \mathsf{z}_2) + \mathsf{z}_3 = 4\mathsf{z}_1 + 2\mathsf{z}_2 + \mathsf{z}_3$$

and

$$\mathsf{Pol} \cdot \lfloor m 1 \lfloor m 2 3 \rfloor \rfloor = 2\mathsf{z}_1 + (2\mathsf{z}_2 + \mathsf{z}_3) = 2\mathsf{z}_1 + 2\mathsf{z}_2 + \mathsf{z}_3.$$

Since $\mathsf{Pol} \cdot \lfloor m \lfloor m 1 2 \rfloor 3 \rfloor - \mathsf{Pol} \cdot \lfloor m 1 \lfloor m 2 3 \rfloor \rfloor = 2\mathsf{z}_1$, this $\mathbb{K}, \mathrm{Mon} \cdot \mathsf{Z}_{\{1,2,3\}}$-polynomial is positive on N.

In the same way, we have $\mathsf{Pol} \cdot \lfloor m' 1 \lfloor m' 2 3 \rfloor \rfloor - \mathsf{Pol} \cdot \lfloor m' \lfloor m' 1 2 \rfloor 3 \rfloor = 2\mathsf{z}_3$, so that the same property holds.

Exercise ●●●○○

Let the signature $\mathcal{S} := (\{f, m\}, \mathrm{ar})$ such that $\mathrm{ar}{\cdot}f = 1$ and $\mathrm{ar}{\cdot}m = 2$.
Let the TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}_{\mathbb{N}}, \to)$ such that $m \lfloor f \mathsf{v}_1 \rfloor \lfloor f \mathsf{v}_2 \rfloor \to f \lfloor m \mathsf{v}_1 \mathsf{v}_2 \rfloor$ and $m \lfloor m \mathsf{v}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \to m \mathsf{v}_1 \lfloor m \mathsf{v}_2 \mathsf{v}_3 \rfloor$.
By using the polynomial interpretation method, show that $\mathcal{T}$ is terminating.

Exercise ●●●○○

By using the polynomial interpretation method, show that the TRS NatAdd is terminating.

Exercise ●●●○○

Let V be the set of variables $\mathbb{N} \setminus \{0\}$ and $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_2, \mathsf{c}_3\}$.
Let the TRS $\mathsf{Motz} := (\mathcal{S}, V, \to)$ such that $\to$ satisfies

$$\mathsf{c}_2 \lfloor \mathsf{c}_2 12 \rfloor 3 \to \mathsf{c}_2 1 \lfloor \mathsf{c}_2 23 \rfloor, \quad \mathsf{c}_3 \lfloor \mathsf{c}_2 12 \rfloor 34 \to \mathsf{c}_2 1 \lfloor \mathsf{c}_3 234 \rfloor,$$

$$\mathsf{c}_2 \lfloor \mathsf{c}_3 123 \rfloor 4 \to \mathsf{c}_3 12 \lfloor \mathsf{c}_2 34 \rfloor, \quad \mathsf{c}_3 \lfloor \mathsf{c}_3 123 \rfloor 45 \to \mathsf{c}_3 12 \lfloor \mathsf{c}_3 345 \rfloor,$$

By using the polynomial interpretation method, show that Motz is terminating.

Exercise ●●●○○

By using the polynomial interpretation method, show that the TRS Schr is terminating.

7.5. Syntactic method

Definition

Let $\mathcal{S}$ be a signature and V be a set of variables. A binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ is a *simplification relation* if

- $\rightsquigarrow$ is acyclic;
- $\rightsquigarrow$ is compatible from factors;
- for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ and any position $w \in \mathrm{P}{\cdot}\mathfrak{t} \setminus \{\epsilon\}$, $\mathfrak{t} \rightsquigarrow \mathfrak{t}{\cdot}w$.

Theorem [Simplification relations and termination]

For any finite signature $\mathcal{S}$ and any finite set of variables V, if $\rightsquigarrow$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$, then the ARS $(\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}, \rightsquigarrow)$ is terminating.

Theorem [Simplification relations and termination] implies that any simplification relation is a **reduction relation**. The **converse** of this property is **false**.

Exercise ●●●○○

Give an example of a finite signature $\mathcal{S}$, of a finite set of variables V, and of a reduction relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ such that $\rightsquigarrow$ is nonempty and $\rightsquigarrow$ is not a simplification relation.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\succ$ be a binary relation on C.

An $\mathcal{S}, \mathsf{V}, \succ$*-weight function* is a function $\omega : C \sqcup \mathsf{V} \to \mathbb{R}_{\geqslant 0}$ such that

1. all variables of V are sent via ω to the same nonnegative real number ω_{V};
2. for all $c \in \mathcal{S}\cdot 0$, $\omega \cdot c \geqslant \omega_{\mathsf{V}}$;
3. for all $c \in \mathcal{S}\cdot 1$, $\omega \cdot c = 0$ implies that for any $c' \in C \setminus \{c\}$, $c \succ c'$.

The ω*-weight* of a $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ is the nonnegative real number

$$\omega \cdot \mathfrak{t} := \sum_{i \in [\ell \cdot \mathfrak{t}]} \omega \cdot \lfloor \mathrm{w} \cdot \mathfrak{t} \cdot i \rfloor.$$

Example

Let $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_4\}$. Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $\mathsf{c}_1 \succ \mathsf{c}_0$, $\mathsf{c}_1 \succ \mathsf{c}_2$, $\mathsf{c}_1 \succ \mathsf{c}_4$, and $\mathsf{c}_0 \succ \mathsf{c}_2$.

Let the $\mathcal{S}, \mathsf{V}_{\mathbb{N}}, \succ$-weight function ω defined by $\omega \cdot v := 1$ for any $v \in \mathsf{V}_{\mathbb{N}}$, $\omega \cdot \mathsf{c}_0 := \frac{3}{2}$, $\omega \cdot \mathsf{c}_1 := 0$, $\omega \cdot \mathsf{c}_2 := \frac{1}{2}$, $\omega \cdot \mathsf{c}_4 := 2$.

We have

$$\omega \cdot \mathsf{c}_4 \mathsf{c}_0 \mathsf{c}_0 \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_1 \rfloor \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor = 2 + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} + 1 + 1 + \frac{1}{2} + 1 = 9.$$

Let $\mathcal{S}$ be a signature, V be a set of variables, $\succ$ be a binary relation on the underlying set of $\mathcal{S}$, and ω be an $\mathcal{S}, \mathsf{V}, \succ$-weight function.

The *$\succ, \omega$-Knuth-Bendix relation* is the binary relation $>_{\mathrm{KB}}^{\succ, \omega}$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ defined recursively as follows. For any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, we have $\mathfrak{t} >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'$ if for all $v \in \mathsf{V}$, $\ell_v \cdot \mathfrak{t} \geqslant \ell_v \cdot \mathfrak{t}'$ and one of the following assertions holds:

- [Weight Case] $\omega \cdot \mathfrak{t} > \omega \cdot \mathfrak{t}'$;
- [Unary Case] $\omega \cdot \mathfrak{t} = \omega \cdot \mathfrak{t}'$, $\mathfrak{t} = c \lfloor c \dots \lfloor c\, v \rfloor \dots \rfloor$ with $c \in \mathcal{S} \cdot 1$, $v \in \mathsf{V}$, $\ell_c \cdot \mathfrak{t} = n$, $n \geqslant 1$, and $\mathfrak{t}' = v$;
- [Precedence Case] $\omega \cdot \mathfrak{t} = \omega \cdot \mathfrak{t}'$, $\mathfrak{t} = c \lfloor \mathfrak{t} \cdot 1 \rfloor \dots \lfloor \mathfrak{t} \cdot n \rfloor$ with $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $\mathfrak{t}' = c' \lfloor \mathfrak{t}' \cdot 1 \rfloor \dots \lfloor \mathfrak{t}' \cdot n' \rfloor$ with $c' \in \mathcal{S} \cdot n'$, $n' \in \mathbb{N}$, and $c \succ c'$;
- [Lexicographic Case] $\omega \cdot \mathfrak{t} = \omega \cdot \mathfrak{t}'$, $\mathfrak{t} = c \lfloor \mathfrak{t} \cdot 1 \rfloor \dots \lfloor \mathfrak{t} \cdot n \rfloor$, $\mathfrak{t}' = c \lfloor \mathfrak{t}' \cdot 1 \rfloor \dots \lfloor \mathfrak{t}' \cdot n \rfloor$ with $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and there is $i \in [n]$ such that $\mathfrak{t} \cdot i >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}' \cdot i$ and for any $j \in [i-1]$, $\mathfrak{t} \cdot j = \mathfrak{t}' \cdot j$.

Proposition [Knuth-Bendix simplification relation]

For any signature $\mathcal{S}$, any set of variables V, any binary relation $\succ$ on $\mathcal{S}$ such that the ARS $(\mathcal{S}, \succ)$ is terminating, and any $\mathcal{S}, \mathsf{V}, \succ$-weight function ω, the binary relation $>_{\mathrm{KB}}^{\succ, \omega}$ is a simplification relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

This result is due to [D. Knuth, P. Bendix, Simple Word Problems in Universal Algebras, 1970] and [N. Dershowitz, Orderings for Term-Rewriting Systems, 1982].

Examples

Let the signature $\mathcal{S} := (\{z, s, s'.a\}, \mathrm{ar})$ such that $\mathrm{ar}{\cdot}z = 0$, $\mathrm{ar}{\cdot}s = 1$, $\mathrm{ar}{\cdot}s' = 1$, and $\mathrm{ar}{\cdot}a = 2$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $a \succ s$, $s \succ z$, $s' \succ z$, $s' \succ s$, and $s' \succ a$.

Let ω be the $\mathcal{S}, \mathsf{V}, \succ$-weight function defined by $\omega_{\mathsf{V}} := 1$, $\omega{\cdot}z := 1$, $\omega{\cdot}s := 1$, $\omega{\cdot}s' := 0$, and $\omega{\cdot}a = 2$.

Let the $\mathcal{S}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := a\,\underline{a z \mathsf{v}_1}\,\underline{s\,\underline{a \mathsf{v}_2 \underline{s z}}}$$

- Let $\mathfrak{t}' := a \mathsf{v}_1 \mathsf{v}_2$. We have $\mathfrak{t} >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'$ by applying the Weight Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathfrak{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathfrak{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathfrak{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_2}{\cdot}\mathfrak{t}'$, and $\omega{\cdot}\mathfrak{t} = 12 > 4 = \omega{\cdot}\mathfrak{t}'$.
- Let $\mathfrak{t}' := s\,\underline{a\,\underline{a \underline{s z} \mathsf{v}_1}\,\underline{a z z}}$. We have $\mathfrak{t} >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'$ by applying the Precedence Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathfrak{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathfrak{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}{\cdot}\mathfrak{t}'$, $\omega{\cdot}\mathfrak{t} = 12 = \omega{\cdot}\mathfrak{t}'$, and $a \succ s$.
- Let $\mathfrak{t}' := a\,\underline{a z \mathsf{v}_1}\,\underline{s\,\underline{s \underline{a z z}}}$. We have $\mathfrak{t} >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'$ by applying the Lexicographic Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathfrak{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathfrak{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}{\cdot}\mathfrak{t}'$, $\omega{\cdot}\mathfrak{t} = 12 = \omega{\cdot}\mathfrak{t}'$, $\mathfrak{t}{\cdot}1 = \mathfrak{t}'{\cdot}1$, and $\mathfrak{t}{\cdot}1 >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'{\cdot}1$.

Moreover, by setting $\mathfrak{t} := s'\,\underline{s'\,\underline{s' \mathsf{v}_1}}$ and $\mathfrak{t}' := \mathsf{v}_1$, we have $\mathfrak{t} >_{\mathrm{KB}}^{\succ, \omega} \mathfrak{t}'$ by applying the Unary Case: $\ell_{\mathsf{v}_1}{\cdot}\mathfrak{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathfrak{t}'$, $\ell_s{\cdot}\mathfrak{t} \geqslant 1$, and $\omega{\cdot}\mathfrak{t} = 0 = \omega{\cdot}\mathfrak{t}'$.

Example

Let $\mathcal{T} := (\mathcal{S}, \{\mathsf{v}\}, \to)$ be the TRS such that $\mathcal{S} := (\{f, g\}, \mathrm{ar})$ where $\mathrm{ar} \cdot f = 1$ and $\mathrm{ar} \cdot g = 1$, and $g \lfloor g \mathsf{v} \rfloor \to f \mathsf{v}$ and $f \lfloor g \mathsf{v} \rfloor \to g \lfloor f \mathsf{v} \rfloor$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $f \succ g$.

Let ω be the $\mathcal{S}, \{\mathsf{v}\}, \succ$-weight function defined by $\omega \cdot \mathsf{v} := 1$, $\omega \cdot f := 1$, and $\omega \cdot g := 1$.

It can be easily checked that $g \lfloor g \mathsf{v} \rfloor >_{\mathrm{KB}}^{\succ, \omega} f \mathsf{v}$ and $f \lfloor g \mathsf{v} \rfloor >_{\mathrm{KB}}^{\succ, \omega} g \lfloor f \mathsf{v} \rfloor$. Therefore, $\mathcal{T}$ is terminating.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be the TRS such that $\mathcal{S} := (\{s, a\}, \mathrm{ar})$ where $\mathrm{ar} \cdot s = 1$ and $\mathrm{ar} \cdot a = 2$, $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4\}$, and $a \lfloor s \mathsf{v}_1 \rfloor \lfloor a \mathsf{v}_2 \mathsf{v}_3 \rfloor \to a \mathsf{v}_1 \lfloor a \lfloor s \lfloor s \mathsf{v}_2 \rfloor \rfloor \mathsf{v}_3 \rfloor$ and $a \lfloor s \mathsf{v}_1 \rfloor \lfloor a \mathsf{v}_2 \lfloor a \mathsf{v}_3 \mathsf{v}_4 \rfloor \rfloor \to a \mathsf{v}_1 \lfloor a \mathsf{v}_3 \lfloor a \mathsf{v}_2 \mathsf{v}_4 \rfloor \rfloor$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $s \succ a$.

Let ω be the $\mathcal{S}, \mathsf{V}, \succ$-weight function defined by $\omega_{\mathsf{v}} := 1$, $\omega \cdot s := 0$, and $\omega \cdot a := 0$.

It can be easily checked that $a \lfloor s \mathsf{v}_1 \rfloor \lfloor a \mathsf{v}_2 \mathsf{v}_3 \rfloor >_{\mathrm{KB}}^{\succ, \omega} a \mathsf{v}_1 \lfloor a \lfloor s \lfloor s \mathsf{v}_2 \rfloor \rfloor \mathsf{v}_3 \rfloor$ and $a \lfloor s \mathsf{v}_1 \rfloor \lfloor a \mathsf{v}_2 \lfloor a \mathsf{v}_3 \mathsf{v}_4 \rfloor \rfloor >_{\mathrm{KB}}^{\succ, \omega} a \mathsf{v}_1 \lfloor a \mathsf{v}_3 \lfloor a \mathsf{v}_2 \mathsf{v}_4 \rfloor \rfloor$. Therefore, $\mathcal{T}$ is terminating.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\succ$ be a binary relation on the underlying set of $\mathcal{S}$.

The *$\succ$-lexicographic path relation* is the binary relation $>_{\mathrm{LP}}^{\succ}$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ defined recursively as follows. For any $\mathcal{S},\mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, we have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ if one of the following assertion holds:

- [Subterm Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, and there exists $i \in [n]$ such that $\mathfrak{t}{\cdot}i = \mathfrak{t}'$ or $\mathfrak{t}{\cdot}i >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$;
- [Precedence Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $\mathfrak{t}' = c' \llcorner \mathfrak{t}'{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}'{\cdot}n' \lrcorner$ with $c' \in \mathcal{S}{\cdot}n'$, $n' \in \mathbb{N}$, $c \succ c'$, and $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}i'$ for any $i' \in [n']$;
- [Lexicographic Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$, $\mathfrak{t}' = c \llcorner \mathfrak{t}'{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}'{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}i$ for all $i \in [n]$, and there is $j \in [n]$ such that $\mathfrak{t}{\cdot}j >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}j$ and for any $j' \in [i-1]$, $\mathfrak{t}{\cdot}j' = \mathfrak{t}'{\cdot}j'$.

Proposition [Lexicographic path simplification relation]

For any signature $\mathcal{S}$, any set of variables V, and any binary relation $\succ$ on $\mathcal{S}$ such that the ARS $(\mathcal{S}, \succ)$ is terminating, the binary relation $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

This result is due to [S. Kamin, J.-J. Levy, Two Generalizations of the Recursive Path Ordering, 1980].

Examples

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ such that $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $a \succ s$ and $s \succ z$.

Let the $\mathcal{S}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t} := a \lfloor s \mathsf{v}_1 \rfloor \lfloor a z \lfloor s \mathsf{v}_2 \rfloor \rfloor$.

- Let $\mathfrak{t}' := s \mathsf{v}_2$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying recursively the Subterm Case: we have $\mathfrak{t} \cdot 22 = \mathfrak{t}'$, which implies $\mathfrak{t} \cdot 2 >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$, which implies $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$.
- Let $\mathfrak{t}' := s z$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying the Precedence Case: we have $a \succ s$ and $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 1$.
- Let $\mathfrak{t}' := a \lfloor s \mathsf{v}_1 \rfloor \lfloor s \mathsf{v}_2 \rfloor$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying the Lexicographic Case: we have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 1$, $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 2$, $\mathfrak{t} \cdot 1 = \mathfrak{t}' \cdot 1$, and $\mathfrak{t} \cdot 2 >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 2$.

Example

Let $\mathsf{Ack} := (\mathcal{S}, \mathsf{V}, \to)$ be the TRS such that $\mathcal{S} := (\{z, s, k\}, \mathrm{ar})$ with $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot k = 2$, $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2\}$, $k z \mathsf{v}_1 \to s \mathsf{v}_1$, $k \lfloor s \mathsf{v}_1 \rfloor z \to k \mathsf{v}_1 \lfloor s z \rfloor$, and $k \lfloor s \mathsf{v}_1 \rfloor \lfloor s \mathsf{v}_2 \rfloor \to k \mathsf{v}_1 \lfloor k \lfloor s \mathsf{v}_1 \rfloor \mathsf{v}_2 \rfloor$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $k \succ s$.

It can be easily checked that $k z \mathsf{v}_1 >_{\mathrm{LP}}^{\succ} s \mathsf{v}_1$, $k \lfloor s \mathsf{v}_1 \rfloor z >_{\mathrm{LP}}^{\succ} k \mathsf{v}_1 \lfloor s z \rfloor$, and $k \lfloor s \mathsf{v}_1 \rfloor \lfloor s \mathsf{v}_2 \rfloor >_{\mathrm{LP}}^{\succ} k \mathsf{v}_1 \lfloor k \lfloor s \mathsf{v}_1 \rfloor \mathsf{v}_2 \rfloor$. Therefore, Ack is terminating.

Exercise ●●○○○

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ such that $\mathcal{S} := (\{f, g, a\}, \mathrm{ar})$ with $\mathrm{ar} \cdot f = 1$, $\mathrm{ar} \cdot g = 1$, and $\mathrm{ar} \cdot a = 2$, $\mathsf{V} := \{\mathsf{v}\}$, and $f \lfloor g \mathsf{v} \rfloor \to a \lfloor g \lfloor f \mathsf{v} \rfloor \rfloor \mathsf{v}$.

Show that there exists a binary relation $\succ$ on the underlying set of $\mathcal{S}$ such that $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

Exercise ●●●○○

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ such that $\mathcal{S} := (\{f, g, h\}, \mathrm{ar})$ with $\mathrm{ar} \cdot f = 1$, $\mathrm{ar} \cdot g = 1$, and $\mathrm{ar} \cdot h = 1$, $\mathsf{V} := \{\mathsf{v}\}$, $g \lfloor f \mathsf{v} \rfloor \to f \lfloor h \mathsf{v} \rfloor$, and $hv \to gv$.

1. Show that the termination of $\mathcal{T}$ cannot be proven by the existence of a binary relation $\succ$ on the underlying set of $\mathcal{S}$ and an $\mathcal{S}, \mathsf{V}, \succ$-weight function ω so that $>_{\mathrm{KB}}^{\succ, \omega}$ is a simplification relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.
2. Show that the termination of $\mathcal{T}$ cannot be proven by the existence of a binary relation $\succ$ on the underlying set of $\mathcal{S}$ so that $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.
3. Show that the termination of $\mathcal{T}$ can be proven by using the polynomial interpretation method.

8. Confluence

8.1. Unification

Let $\mathcal{S}$ be a signature and V be a set of variables.

Given two $\mathcal{S}, \mathsf{V}$-substitutions σ and σ', σ is *more general* (or *less specific*) than σ' if there exists an $\mathcal{S}, \mathsf{V}$-substitution σ'' such that $\sigma' = \sigma'' \circ \sigma$. This property is denoted by $\sigma \leqslant_{\mathrm{g}} \sigma'$.

Examples

- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_3), (\mathsf{v}_2, \mathsf{c}_0)\}]$ and $\sigma' := [\{(\mathsf{v}_1, \mathsf{c}_1\underline{\mathsf{c}_2\mathsf{c}_0\mathsf{v}_1}), (\mathsf{v}_2, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1)\}]$. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ because $[\{(\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1)\}] \circ \sigma = \sigma'$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_0)\}]$ and $\sigma' := [\{(\mathsf{v}_2, \mathsf{c}_1\mathsf{v}_3)\}]$. We have $\sigma \not\leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \not\leqslant_{\mathrm{g}} \sigma$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_2\mathsf{v}_1\mathsf{v}_3)\}]$ and $\sigma' := [\{(\mathsf{v}_1, \mathsf{c}_2\mathsf{v}_3\mathsf{v}_1)\}]$. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \leqslant_{\mathrm{g}} \sigma$.

Proposition [Generality relation on $\mathcal{S}, \mathsf{V}$-substitutions]

Let $\mathcal{S}$ be a signature and V be a set of variables.

- The binary relation $\leqslant_{\mathrm{g}}$ is a preorder.
- Let σ and σ' be two $\mathcal{S}, \mathsf{V}$-substitutions. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \leqslant_{\mathrm{g}} \sigma$ iff there exists a **renaming** $\mathcal{S}, \mathsf{V}$-substitution σ'' such that $\sigma' = \sigma'' \circ \sigma$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. If there exists an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\overline{\sigma} \cdot \mathfrak{t} = \overline{\sigma} \cdot \mathfrak{t}'$, then

- the $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *unifiable*. This property is denoted by $\mathfrak{t} \sim_{\mathrm{u}} \mathfrak{t}'$;
- the $\mathcal{S}, \mathsf{V}$-substitution σ is a *unifier* of $\mathfrak{t}$ and $\mathfrak{t}'$;
- when for any unifier σ' of $\mathfrak{t}$ and $\mathfrak{t}'$, $\sigma \leqslant_{\mathrm{g}} \sigma'$, σ is a *most general unifier* (*MGU*) of $\mathfrak{t}$ and $\mathfrak{t}'$;
- when σ is an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$, the $\mathcal{S}, \mathsf{V}$-term $\overline{\sigma} \cdot \mathfrak{t} = \overline{\sigma} \cdot \mathfrak{t}'$ is a *most general common instance* (*MGCI*) of $\mathfrak{t}$ and $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor$ and $\mathfrak{t}' := \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3$.

- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma_1 := [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor)\}]$.
  Since $\overline{\sigma_1} \cdot \mathfrak{t} = \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \rfloor = \overline{\sigma_1} \cdot \mathfrak{t}'$, σ_1 is a unifier of $\mathfrak{t}$ and $\mathfrak{t}'$.
  Moreover, σ_1 is an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$, and $\mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \rfloor$ is an MGCI of $\mathfrak{t}$ and $\mathfrak{t}'$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma_2 := [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor)\}]$.
  Since $\overline{\sigma_2} \cdot \mathfrak{t} = \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \rfloor = \overline{\sigma_2} \cdot \mathfrak{t}'$, σ_2 is a unifier of $\mathfrak{t}$ and $\mathfrak{t}'$.
  Moreover, since $\sigma_2 = [\{(\mathsf{v}_2, \mathsf{c}_0)\}] \circ \sigma_1$, σ_2 is not an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

An $\mathcal{S}, \mathsf{V}$-substitution σ is *idempotent* if $\sigma \circ \sigma = \sigma$.

Given an $\mathcal{S}, \mathsf{V}$-substitution σ, the *set of variables* of σ is the set

$$\mathrm{Vars} \cdot \sigma := \bigcup_{v \in \mathrm{Dom} \cdot \sigma} \mathrm{Vars} \cdot \lfloor \sigma \cdot v \rfloor.$$

Proposition [Idempotent $\mathcal{S}, \mathsf{V}$-substitutions]

Let $\mathcal{S}$ be a signature and V be a set of variables. An $\mathcal{S}, \mathsf{V}$-substitution σ is idempotent iff the sets $\mathrm{Dom} \cdot \sigma$ and $\mathrm{Vars} \cdot \sigma$ are disjoint.

Theorem [Idempotent MGUs]

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms. If $\mathfrak{t}$ and $\mathfrak{t}'$ are unifiable, then there exists an idempotent MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

For any $S \in \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}\rfloor^2$, let $\mathrm{Vars}\cdot S := \bigcup_{(\mathfrak{t},\mathfrak{t}') \in S} \mathrm{Vars}\cdot\mathfrak{t} \cup \mathrm{Vars}\cdot\mathfrak{t}'$. Moreover, for any $S \in \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}\rfloor^2$ and any $\mathcal{S},\mathsf{V}$-substitution σ, let $\overline{\sigma}\cdot S := \{(\overline{\sigma}\cdot\mathfrak{t}, \overline{\sigma}\cdot\mathfrak{t}') : (\mathfrak{t},\mathfrak{t}') \in S\}$.

Let the ARS $\mathsf{Unification}_{\mathcal{S},\mathsf{V}} := \left(\{\mathsf{Fail}\} \cup \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}\rfloor^2, \Rightarrow\right)$ such that

1. [Simplification] $\{(v, v)\} \sqcup S \;\Rightarrow\; S$ if $v \in \mathsf{V}$;
2. [Decomposition] $\{(c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c\,\mathfrak{t}'_1 \dots \mathfrak{t}'_n)\} \sqcup S \;\Rightarrow\; \{(\mathfrak{t}_1, \mathfrak{t}'_1), \dots, (\mathfrak{t}_n, \mathfrak{t}'_n)\} \cup S$;
3. [Orientation] $\{(\mathfrak{t}, v)\} \sqcup S \;\Rightarrow\; \{(v, \mathfrak{t})\} \cup S$ if $\mathfrak{t} \notin \mathsf{V}$ and $v \in \mathsf{V}$;
4. [Variable elimination] $\{(v, \mathfrak{t})\} \sqcup S \;\Rightarrow\; \{(v, \mathfrak{t})\} \cup \overline{[\{(v, \mathfrak{t})\}]}\cdot S$ if $v \notin \mathrm{Vars}\cdot\mathfrak{t}$ and $v \in \mathrm{Vars}\cdot S$;
5. [Recursive variable occurrence] $\{(v, \mathfrak{t})\} \sqcup S \;\Rightarrow\; \mathsf{Fail}$ if $\mathfrak{t} \notin \mathsf{V}$ and $v \in \mathrm{Vars}\cdot\mathfrak{t}$;
6. [Constant clash] $\{(c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c'\,\mathfrak{t}'_1 \dots \mathfrak{t}'_{n'})\} \sqcup S \;\Rightarrow\; \mathsf{Fail}$ if $c \neq c'$.

This ARS leads to the **Martelli-Montanari unification Algorithm** [A. Martelli, U. Montanari, Unification in linear time and space: a structured presentation, 1976].

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms.

The ARS $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$ is used by computing a **normal form** of $\{(\mathfrak{t}, \mathfrak{t}')\}$ and, when this normal form is a set S, by considering the $\mathcal{S}, \mathsf{V}$-**substitution** $[S]$ **specified** by S.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t} := \mathsf{c}_2\mathsf{v}_1\lfloor\mathsf{c}_2\mathsf{c}_0\mathsf{v}_1\rfloor$ and $\mathfrak{t}' := \mathsf{c}_2\lfloor\mathsf{c}_1\mathsf{v}_2\rfloor\mathsf{v}_3$. In $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2},\mathsf{V}_{\mathbb{N}}}$,

$$\left\{\left(\mathfrak{t}, \mathfrak{t}'\right)\right\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{c}_2\mathsf{c}_0\mathsf{v}_1, \mathsf{v}_3)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\lfloor\mathsf{c}_1\mathsf{v}_2\rfloor)\} =: \sigma.$$

We can check that the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution σ is an idempotent MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t} := \mathsf{c}_3\mathsf{v}_1\lfloor\mathsf{c}_2\mathsf{v}_2\mathsf{c}_0\rfloor\mathsf{v}_1$ and $\mathfrak{t}' := \mathsf{c}_3\lfloor\mathsf{c}_1\mathsf{v}_2\rfloor\mathsf{v}_3\mathsf{v}_2$. In $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2},\mathsf{V}_{\mathbb{N}}}$,

$$\left\{\left(\mathfrak{t}, \mathfrak{t}'\right)\right\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{c}_2\mathsf{v}_2\mathsf{c}_0, \mathsf{v}_3), (\mathsf{v}_1, \mathsf{v}_2)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{c}_2\mathsf{v}_2\mathsf{c}_0, \mathsf{v}_3), (\mathsf{c}_1\mathsf{v}_2, \mathsf{v}_2)\}$$
$$\Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_2), (\mathsf{c}_2\mathsf{v}_2\mathsf{c}_0, \mathsf{v}_3), (\mathsf{v}_2, \mathsf{c}_1\mathsf{v}_2)\} \Rightarrow \mathsf{Fail}.$$

We can check that $\mathfrak{t}$ and $\mathfrak{t}'$ are not unifiable.

Due to the following results, the ARS $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$, where $\mathcal{S}$ is any signature and V is any set of variables, computes exactly what it is designed to compute.

Theorem [Termination of $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$]

For any signature $\mathcal{S}$ and set of variables V, the ARS $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$ is terminating.

Exercise ●●○○○

Show that $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$ is not confluent, where $\mathcal{S}$ is a signature and V is a set of variables.

Proposition [Computation of $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$, and $S := \{(\mathfrak{t}, \mathfrak{t}')\}$.

- If $\mathfrak{t} \sim_{\mathrm{u}} \mathfrak{t}'$, then any normal form of S in $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$ is a set S' such that $[S']$ is an idempotent MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.
- Otherwise, the unique normal form of S in $\mathsf{Unification}_{\mathcal{S},\mathsf{V}}$ is Fail.

Exercise ●○○○○

By using the ARS $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}}$, compute a normal form of $\{(t, t')\}$ where

1. $t := c_3 v_1 \underline{c_2 c_0 v_1} \, \underline{c_1 v_2}$ and $t' := c_3 \underline{c_1 v_2} v_3 v_4$;
2. $t := c_2 v_1 \underline{c_1 v_2}$ and $t' := c_2 \underline{c_2 v_3 v_2} v_1$.

Exercise ●●○○○

Describe the relation between the ARSs $\mathsf{Matching}_{\mathcal{S}, V}$ and $\mathsf{Unification}_{\mathcal{S}, V}$ where $\mathcal{S}$ is a signature and V is a set of variables.

Exercise ●●●○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Let t and t' be two $\mathcal{S}, V$-terms such that $t \sim_{u} t'$. Show that an MGCI of t and t' is a supremum of t and t' for the preorder $\preccurlyeq_{p}$.

8.2. Overlaps and critical data

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{r}$ be a $\mathcal{S}, \mathsf{V}$-term admitting the two decompositions

$$\blacktriangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \sigma_1 = \mathfrak{r} = \blacktriangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}_2 \cdot \sigma_2$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are holed $\mathcal{S}, \mathsf{V}$-terms, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are $\mathcal{S}, \mathsf{V}$-terms, and σ_1 and σ_2 are $\mathcal{S}, \mathsf{V}$-substitutions. Let w_1 (resp. w_2) be the hole position of $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$). Without loss of generality, let us consider that $\ell \cdot w_1 \leqslant \ell \cdot w_2$.

We distinguish the following disjoint and exhaustive three cases:

1. [Horizontal disjunction] This occurs when w_1 is not a prefix of w_2;
2. [Vertical disjunction] This occurs when w_1 is a prefix of w_2, and, by setting u as the word over positive integers such that $w_2 = w_1 . u$, u is not the position of an internal node of $\mathfrak{t}_1$;
3. [Overlap] This occurs when w_1 is a prefix of w_2, and, by setting u as the word over positive integers such that $w_2 = w_1 . u$, u is the position of an internal node of $\mathfrak{t}_1$.

Note that when $w_1 = w_2$, we are in the Vertical disjunction Case if $\mathfrak{t}_1$ is a variable and in the Overlap Case otherwise.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS.

Let $\mathfrak{r}$ be an $\mathcal{S}, \mathsf{V}$-term, and w_1 and w_2 be two positions within $\mathfrak{r}$ such that $\ell \cdot w_1 \leqslant \ell \cdot w_2$, $\mathfrak{r} \Rightarrow_{w_1} \mathfrak{u}_1$, and $\mathfrak{r} \Rightarrow_{w_2} \mathfrak{u}_2$ where $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are two $\mathcal{S}, \mathsf{V}$-terms. By Proposition [Rewrite relation of a TRS],

$$\mathfrak{r} = \blacktriangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \sigma_1 \quad \Rightarrow_{w_1} \quad \blacktriangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}'_1 \cdot \sigma_1 = \mathfrak{u}_1$$

and

$$\mathfrak{r} = \blacktriangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}_2 \cdot \sigma_2 \quad \Rightarrow_{w_2} \quad \blacktriangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}'_2 \cdot \sigma_2 = \mathfrak{u}_2$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are holed $\mathcal{S}, \mathsf{V}$-terms, $\mathfrak{t}_1$, $\mathfrak{t}'_1$, $\mathfrak{t}_2$, and $\mathfrak{t}'_2$ are $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t}_1 \to \mathfrak{t}'_1$ and $\mathfrak{t}_2 \to \mathfrak{t}'_2$, and σ_1 and σ_2 are $\mathcal{S}, \mathsf{V}$-substitutions.

Let us understand what happens in each of the three previous cases.

Let us consider the previous notations and definitions.

In the **Horizontal disjunction Case**, w_1 is not a prefix of w_2.

Therefore, by considering that $\square_1$ and $\square_2$ are two variables which do not belong to V, we have

$$\mathfrak{r} = \mathfrak{s}[\{(\square_1, \overline{\sigma_1} \cdot \mathfrak{t}_1), (\square_2, \overline{\sigma_2} \cdot \mathfrak{t}_2)\}]$$

where $\mathfrak{s}$ is an $\mathcal{S}, \mathsf{V} \sqcup \{\square_1, \square_2\}$-term having exactly one occurrence of $\square_1$ and exactly one occurrence of $\square_2$ at respective positions w_1 and w_2.

We have

$$\mathfrak{u}_1 \Rightarrow_{w_2} \mathfrak{u}$$

and

$$\mathfrak{u}_2 \Rightarrow_{w_1} \mathfrak{u}$$

where $\mathfrak{u}$ is the $\mathcal{S}, \mathsf{V}$-term defined by

$$\mathfrak{u} := \mathfrak{s}[\{(\square_1, \overline{\sigma_1} \cdot \mathfrak{t}'_1), (\square_2, \overline{\sigma_2} \cdot \mathfrak{t}'_2)\}].$$

Let us consider the previous notations and definitions.

In the **Vertical disjunction Case**, w_1 is a prefix of w_2, and, by writing $w_2 = w_1 \centerdot u$, u is not the position of an internal node of $\mathfrak{t}_1$.

There exist $v \in \mathsf{V}$ and two words u_0 and p such that $u = u_0 \centerdot p$, $u_0 \in \mathrm{P} \cdot \mathfrak{t}_1$, and $\mathfrak{t}_1 \cdot u_0 = v$.

Moreover, there exist a holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{q}$ and an $\mathcal{S}, \mathsf{V}$-substitution ρ such that

$$\sigma_1 \cdot v = \blacktriangle \cdot \mathfrak{q} \cdot \mathfrak{t}_2 \cdot \rho \quad \Rightarrow_p \quad \blacktriangle \cdot \mathfrak{q} \cdot \mathfrak{t}'_2 \cdot \rho.$$

Let τ be the $\mathcal{S}, \mathsf{V}$-substitution defined by $\tau \cdot v := \blacktriangle \cdot \mathfrak{q} \cdot \mathfrak{t}'_2 \cdot \rho$, and $\tau \cdot v' := \sigma_1 \cdot v'$ for any $v' \in \mathsf{V} \setminus \{v\}$.

Let $n := \ell_v \cdot \mathfrak{t}_1$ (resp. $n' := \ell_v \cdot \mathfrak{t}'_1$) and $\{u_0, \dots, u_{n-1}\}$ (resp. $\{u'_0, \dots, u'_{n'-1}\}$) be the set of positions of the variable v in $\mathfrak{t}_1$ (resp. $\mathfrak{t}'_1$). We have

$$\mathfrak{u}_1 \quad \Rightarrow_{w_1 \centerdot u'_0 \centerdot p} \quad \cdots \quad \Rightarrow_{w_1 \centerdot u'_{n'-1} \centerdot p} \quad \mathfrak{u}$$

and

$$\mathfrak{u}_2 \quad \Rightarrow_{w_1 \centerdot u_1 \centerdot p} \quad \cdots \quad \Rightarrow_{w_1 \centerdot u_{n-1} \centerdot p} \quad \blacktriangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \tau \quad \Rightarrow_{w_1} \quad \mathfrak{u}$$

where $\mathfrak{u}$ is the $\mathcal{S}, \mathsf{V}$-term defined by

$$\mathfrak{u} := \blacktriangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}'_1 \cdot \tau.$$

Let us consider the previous notations and definitions.

In the Overlap Case, w_1 is a prefix of w_2, and, by writing $w_2 = w_1 . u$, u is the position of an internal node of $\mathfrak{t}_1$.

In this case, there is no generic way to exhibit an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{u}$ such that $\mathfrak{u}_1 \Rightarrow^* \mathfrak{u}$ and $\mathfrak{u}_2 \Rightarrow^* \mathfrak{u}$.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS.

We shall describe a necessary and sufficient condition for the fact that $\mathcal{T}$ is locally confluent.

When $\mathcal{T}$ is terminating and $\to$ is finite, this condition leads to an algorithm to decide local confluence of $\mathcal{T}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms such that $\mathrm{Vars} \cdot \mathfrak{t} \cap \mathrm{Vars} \cdot \mathfrak{t}' = \emptyset$. If there exists a position u within $\mathfrak{t}'$ such that $\mathfrak{t}' \cdot u$ is not a leaf, and $\mathfrak{t}' \cdot u$ and $\mathfrak{t}$ are unifiable, then

- $\mathfrak{t}$ *overlaps* $\mathfrak{t}'$;
- u is the *overlapping* position of $\mathfrak{t}$ in $\mathfrak{t}'$;
- if σ is an MGU of $\mathfrak{t}' \cdot u$ and $\mathfrak{t}$, then $\bar{\sigma} \cdot \mathfrak{t}'$ is the *fusion* of $\mathfrak{t}$ at position u into $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}' := \mathsf{c}_{2,0}\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_1\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_2\,\mathsf{v}_3}}\,\mathsf{v}_4$ and $\mathfrak{t} := \mathsf{c}_{2,1}\,\underline{\mathsf{c}_{2,0}\,\mathsf{v}_5\,\mathsf{v}_6}\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_7\,\mathsf{v}_8}$.

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ overlaps $\mathfrak{t}'$ at position $u := 1$.
  The fusion of $\mathfrak{t}$ at position u into $\mathfrak{t}'$ is $\mathsf{c}_{2,0}\,\underline{\mathsf{c}_{2,1}\,\underline{\mathsf{c}_{2,0}\,\mathsf{v}_5\,\mathsf{v}_6}\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_7\,\mathsf{v}_8}}\,\mathsf{v}_4$.

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ overlaps $\mathfrak{t}'$ at position $u := 12$.
  The fusion of $\mathfrak{t}$ at position u into $\mathfrak{t}'$ is $\mathsf{c}_{2,0}\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_1\,\underline{\mathsf{c}_{2,1}\,\underline{\mathsf{c}_{2,0}\,\mathsf{v}_5\,\mathsf{v}_6}\,\underline{\mathsf{c}_{2,1}\,\mathsf{v}_7\,\mathsf{v}_8}}}\,\mathsf{v}_4$.

For any set of variables V and any $k \in \mathbb{N}$, set $\mathsf{V}^{(k)}$ as the set of variables $\{v^{(k)} : v \in \mathsf{V}\}$. We identify $\mathsf{V}^{(0)}$ with V. When $k \geqslant 1$, $\mathsf{V}^{(k)}$ is a **distinct copy** of V. Let also $\mathrm{C}{\cdot}\mathsf{V} := \bigsqcup_{k \in \mathbb{N}} \mathsf{V}^{(k)}$.

Example

Let the set of variables $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2\}$. We have $\mathrm{C}{\cdot}\mathsf{V} = \left\{\mathsf{v}_1^{(0)} = \mathsf{v}_1, \mathsf{v}_2^{(0)} = \mathsf{v}_2, \mathsf{v}_1^{(1)}, \mathsf{v}_2^{(1)}, \mathsf{v}_1^{(2)}, \mathsf{v}_2^{(2)}, \mathsf{v}_1^{(3)}, \mathsf{v}_2^{(3)}, \dots\right\}$.

Let $\mathcal{S}$ be a signature. For any $\mathcal{S}, \mathrm{C}{\cdot}\mathsf{V}$-term $\mathfrak{t}$, the *shift* of $\mathfrak{t}$ is the $\mathcal{S}, \mathrm{C}{\cdot}\mathsf{V}$-term $\mathrm{shift}{\cdot}\mathfrak{t} := \overline{\sigma}{\cdot}\mathfrak{t}$ where σ is any $\mathcal{S}, \mathrm{C}{\cdot}\mathsf{V}$-substitution satisfying $\sigma{\cdot}v^{(k)} = v^{(1+k)}$ for any $v^{(k)} \in \mathrm{C}{\cdot}\mathsf{V}$.

Example

By considering the previous set of variables V, $\mathrm{shift}{\cdot}\mathsf{a}_2 \llcorner \mathsf{a}_1 \mathsf{v}_1 \lrcorner \llcorner \mathsf{a}_2 \mathsf{v}_1^{(3)} \mathsf{v}_2^{(1)} \lrcorner = \mathsf{a}_2 \llcorner \mathsf{a}_1 \mathsf{v}_1^{(1)} \lrcorner \llcorner \mathsf{a}_2 \mathsf{v}_1^{(4)} \mathsf{v}_2^{(2)} \lrcorner$.

The *disjunction* of a pair $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{S}, \mathrm{C}{\cdot}\mathsf{V}$-terms is the pair $\left(\mathfrak{t}, \mathfrak{t}'^{\bullet}\right)$ where $\mathfrak{t}'^{\bullet} := \mathrm{shift}^{\ell}{\cdot}\mathfrak{t}'$ and ℓ is the smallest natural number such that all superscripts of the variables of $\mathfrak{t}$ are smaller than all superscripts of the variables of $\mathfrak{t}'^{\bullet}$.

Example

The disjunction of the pair $\left(\mathsf{a}_2 \mathsf{v}_1 \mathsf{v}_2^{(3)}, \mathsf{a}_3 \mathsf{v}_1^{(1)} \mathsf{v}_2 \mathsf{v}_1^{(5)}\right)$ is $\left(\mathsf{a}_2 \mathsf{v}_1 \mathsf{v}_2^{(3)}, \mathsf{a}_3 \mathsf{v}_1^{(5)} \mathsf{v}_2^{(4)} \mathsf{v}_1^{(9)}\right)$.

For any TRS $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$, let the TRS $\mathrm{C}{\cdot}\mathcal{T} := (\mathcal{S}, \mathrm{C}{\cdot}\mathrm{V}, \to)$ where this second occurrence of $\to$ is the elementary rewrite relation of $\mathcal{T}$ extended to the set $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\lfloor \mathrm{C}{\cdot}\mathrm{V} \rfloor$. In this way, the rewrite relation $\Rightarrow$ of $\mathrm{C}{\cdot}\mathcal{T}$ is the rewrite relation of $\mathcal{T}$ extended to $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\lfloor \mathrm{C}{\cdot}\mathrm{V} \rfloor$.

Definition

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. Let $r_1 := (\mathfrak{t}_1, \mathfrak{t}'_1)$ and $r_2 := (\mathfrak{t}_2, \mathfrak{t}'_2)$ be two rewrite rules of $\mathrm{C}{\cdot}\mathcal{T}$ such that $\mathfrak{t}_2^\bullet$ overlaps $\mathfrak{t}_1$ at an overlapping position u, where $(\mathfrak{t}_1, \mathfrak{t}_2^\bullet)$ is the disjunction of $(\mathfrak{t}_1, \mathfrak{t}_2)$.

- A *critical data* of $\mathcal{T}$ is a triple (r_1, u, r_2) such that $r_1 \neq r_2$ or $u \neq \epsilon$.
- The *critical term* associated with the critical data (r_1, u, r_2) of $\mathcal{T}$ is the fusion of $\mathfrak{t}_2^\bullet$ at position u into $\mathfrak{t}_1$. This critical term is an $\mathcal{S}, \mathrm{C}{\cdot}\mathrm{V}$-term.
- The *critical pair* associated with the critical data (r_1, u, r_2) of $\mathcal{T}$ is the pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of $\mathcal{S}, \mathrm{C}{\cdot}\mathrm{V}$-terms such that, by denoting by $\mathfrak{t}$ the critical term associated with (r_1, u, r_2), $\mathfrak{s}_1$ is obtained by a one-step rewrite at root from $\mathfrak{t}$ by using r_1 in $\mathrm{C}{\cdot}\mathcal{T}$, and $\mathfrak{s}_2$ is obtained by a one-step rewrite at position u from $\mathfrak{t}$ by using r_2 in $\mathrm{C}{\cdot}\mathcal{T}$.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ such that

$$r_1 := \mathsf{c}_{2,0} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_1 \underbracket{\mathsf{c}_{2,1} \mathsf{v}_2 \mathsf{v}_3}} \mathsf{v}_4 \to \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3$$

and

$$r_2 := \mathsf{c}_{2,1} \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1 \mathsf{v}_2} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_3 \mathsf{v}_4} \to \mathsf{c}_1 \mathsf{v}_1.$$

- The triple $(r_1, 1, r_2)$ is a critical data of $\mathcal{T}$.
- The $\mathcal{S}_{\mathbb{N}^2}, \mathrm{C}{\cdot}\mathsf{V}_{\mathbb{N}}$-term

  $$\mathfrak{t} := \mathsf{c}_{2,0} \underbracket{\mathsf{c}_{2,1} \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1^{(1)} \mathsf{v}_2^{(1)}} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_3^{(1)} \mathsf{v}_4^{(1)}}} \mathsf{v}_4$$

  is the critical term associated with this critical data.
- The critical pair associated with this critical data is

  $$\left(\mathsf{c}_3 \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1^{(1)} \mathsf{v}_2^{(1)}} \mathsf{v}_3^{(1)} \mathsf{v}_4^{(1)}, \mathsf{c}_{2,0} \underbracket{\mathsf{c}_1 \mathsf{v}_1^{(1)}} \mathsf{v}_4\right).$$

A critical data (r_1, u, r_2) of a TRS $\mathcal{T}$ is *joinable* if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are joinable in $\mathrm{C}{\cdot}\mathcal{T}$, where $(\mathfrak{s}_1, \mathfrak{s}_2)$ is the critical pair associated with (r_1, u, r_2).

Theorem [Critical pairs and local confluence]

A TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ is locally confluent iff all critical data of $\mathcal{T}$ are joinable.

This leads to the following **algorithm** to prove that a TRS $\mathcal{T}$ having a finite elementary rewrite relation is confluent:

1. Prove that $\mathcal{T}$ is terminating (for instance, by one of the methods seen in the previous chapter).
2. List all critical data of $\mathcal{T}$.
3. For each critical pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of these critical data, compute the future of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathrm{C}{\cdot}\mathcal{T}$ and exhibit a common element.

By Theorems [Newman's Lemma] and [Critical pairs and local confluence], these steps show the confluence of $\mathcal{T}$.

The fact that the elementary rewrite relation of $\mathcal{T}$ is finite implies that $\mathcal{T}$ admits finitely many critical data. Moreover, Step 1. implies that the computation of the future of any elements of a critical pair ends. Therefore, the previous steps form an algorithm.

Example

Let $\mathsf{DA} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing two binary constants a and b, and $\to$ is defined through the four rewrite rules

- $r_1 := a \lfloor a12 \rfloor 3 \to a1 \lfloor a23 \rfloor$;
- $r_2 := b \lfloor a12 \rfloor 3 \to a1 \lfloor b23 \rfloor$;
- $r_3 := b1 \lfloor a23 \rfloor \to b \lfloor b12 \rfloor 3$;
- $r_4 := b1 \lfloor b2 \lfloor b34 \rfloor \rfloor \to a \lfloor b1 \lfloor b23 \rfloor \rfloor 4$.

This TRS is introduced in [S. Giraudo, Combinatorial operads from monoids, 2015].

We assume that we have proven that DA is terminating.

This TRS contains the following critical data: $(r_1, 1, r_1)$, $(r_2, 1, r_1)$, (r_2, ϵ, r_3), (r_2, ϵ, r_4), $(r_3, 2, r_1)$, (r_3, ϵ, r_2), (r_4, ϵ, r_2), $(r_4, 2, r_2)$, $(r_4, 22, r_2)$, $(r_4, 22, r_3)$, $(r_4, 2, r_4)$, and $(r_4, 22, r_4)$.

The critical data $(r_4, 2, r_4)$ has $b1 \lfloor b1^{(1)} \lfloor b2^{(1)} \lfloor b3^{(1)} 4^{(1)} \rfloor \rfloor \rfloor$ as critical term and is joinable since

$$b1 \lfloor b1^{(1)} \lfloor b2^{(1)} \lfloor b3^{(1)} 4^{(1)} \rfloor \rfloor \rfloor \Rightarrow a \lfloor b1 \lfloor b1^{(1)} 2^{(1)} \rfloor \rfloor \lfloor b3^{(1)} 4^{(1)} \rfloor =: \mathfrak{s}$$

and

$$b1 \lfloor b1^{(1)} \lfloor b2^{(1)} \lfloor b3^{(1)} 4^{(1)} \rfloor \rfloor \rfloor \Rightarrow b1 \lfloor a \lfloor b1^{(1)} \lfloor b2^{(1)} 3^{(1)} \rfloor \rfloor 4^{(1)} \rfloor \Rightarrow b \lfloor b1 \lfloor b1^{(1)} \lfloor b2^{(1)} 3^{(1)} \rfloor \rfloor 4^{(1)} \rfloor \Rightarrow b \lfloor a \lfloor b1 \lfloor b1^{(1)} 2^{(1)} \rfloor \rfloor 3^{(1)} \rfloor 4^{(1)} \Rightarrow \mathfrak{s}.$$

Exercise ●●○○○

Show that all other critical data of DA are joinable.

Exercise ●○○○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing two unary constants f and g, and $\to$ is defined through the two rewrite rules $f \llcorner g \llcorner f 1 \lrcorner \lrcorner \to 1$ and $f \llcorner g 1 \lrcorner \to g \llcorner f 1 \lrcorner$. List all critical data of $\mathcal{T}$.

Exercise ●●○○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant u, a unary constant i, and a binary constant m, and $\to$ is defined through the three rewrite rules $mu1 \to 1$, $m \llcorner i1 \lrcorner 1 \to u$, $m \llcorner m12 \lrcorner 3 \to m1 \llcorner m23 \lrcorner$.

1. List all critical data of $\mathcal{T}$.
2. Show that there are some critical data of $\mathcal{T}$ which are not joinable.

Exercise ●●●○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant e and two binary constants m and s, and $\to$ is defined through the seven rewrite rules $m1e \to 1$, $me1 \to 1$, $s1e \to 1$, $se1 \to e$, $s11 \to e$, $s \llcorner m12 \lrcorner 3 \to m \llcorner s13 \lrcorner \llcorner s2 \llcorner s31 \lrcorner \lrcorner$, and $s1 \llcorner m23 \lrcorner \to s \llcorner s12 \lrcorner 3$.

Apply the previously described algorithm to show that $\mathcal{T}$ is confluent.

8.3. Orthogonality

A critical data (r_1, u, r_2) of a TRS $\mathcal{T}$ is *trivial* if $\mathfrak{s}_1 = \mathfrak{s}_2$ where $(\mathfrak{s}_1, \mathfrak{s}_2)$ is the critical pair associated with (r_1, u, r_2).

Definition

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \rightarrow)$ be a TRS. When

- all left-hand sides of rewrite rules of $\mathcal{T}$ are linear, $\mathcal{T}$ is *left-linear*;
- $\mathcal{T}$ admits no critical data, $\mathcal{T}$ is *non-overlapping*;
- all critical data of $\mathcal{T}$ are trivial, $\mathcal{T}$ is *weakly non-overlapping*;
- $\mathcal{T}$ is (weakly) non-overlapping and left-linear, $\mathcal{T}$ is *(weakly) orthogonal*.

Examples

Let $\mathcal{S}$ be the signature containing two nullary constants t and f, and one binary constant o.

- Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \rightarrow)$ be the TRS such that $\rightarrow$ is defined through the three rewrite rules $r_1 := ot1 \rightarrow t$, $r_2 := o1t \rightarrow t$, and $off \rightarrow f$. The critical data (r_1, ϵ, r_2) is trivial and $\mathcal{T}$ is **weakly orthogonal**.
- Let $\mathcal{T}' := (\mathcal{S}, \mathbb{N}, \rightarrow')$ be the TRS such that $\rightarrow'$ is defined through the two rewrite rules $ot1 \rightarrow' t$ and $of1 \rightarrow' 1$. The TRS $\mathcal{T}'$ is **orthogonal**.

Theorem [Confluence of weakly orthogonal TRSs

Any weakly orthogonal TRS is confluent.

This result is important to establish the confluence of TRSs which are **not terminating**.

The condition to be **left-linear** for a weakly orthogonal TRS is **necessary**.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing three nullary constants i, t, and f, a unary constant s, and a binary constant e, and $\to$ is defined through the three rewrite rules $i \to si$, $e11 \to t$, and $e1\llcorner s1 \lrcorner \to f$.

This TRS $\mathcal{T}$ is non-overlapping but is not left-linear.

Since

$$eii \Rightarrow t \quad \text{and} \quad eii \Rightarrow ei\llcorner si \lrcorner \Rightarrow f,$$

the $\mathcal{S}, \mathbb{N}$-term eii has two normal forms. Therefore, $\mathcal{T}$ is not confluent.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant a, a unary constant k, and binary constant f, and $\to$ is defined through the three rewrite rules $r_1 := fa1 \to k1$, $r_2 := f1a \to k1$, and $k1 \to fa1$.

First, $\mathcal{T}$ is not terminating because

$$k1 \Rightarrow fa1 \Rightarrow k1 \Rightarrow fa1 \Rightarrow \cdots$$

leads to an infinite rewrite sequence in $\mathcal{T}$.

Second, all critical data of $\mathcal{T}$ are trivial. Indeed, the only critical data of $\mathcal{T}$ are (r_1, ϵ, r_2) and (r_2, ϵ, r_1). For both these critical data, the associated critical term is faa, and the associated critical pair is (ka, ka).

Therefore, by Theorem [Confluence of weakly orthogonal TRSs], $\mathcal{T}$ is confluent.

8.4. Completion

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$.

Let the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow} := \left(\{\mathsf{Fail}\} \cup \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor\rfloor^2, \Rightarrow\right)$ such that, for any $S \in \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor\rfloor^2$,

1. **[Critical pair join]**

$$S \Rightarrow S \sqcup \{(\mathfrak{t}_1, \mathfrak{t}_2)\}$$

   if there exists a critical pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of a critical data of the TRS $\mathcal{T} := (\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$, such that $\mathfrak{t}_1$ is a normal form of $\mathfrak{s}_1$ in $\mathcal{T}$ and $\mathfrak{t}_2$ is a normal form of $\mathfrak{s}_2$ in $\mathcal{T}$, and $\mathfrak{t}_1 \neq \mathfrak{t}_2$.

2. **[Reduction clash]**

$$S \Rightarrow \mathsf{Fail}$$

   if there exists $(\mathfrak{t}_1, \mathfrak{t}_2) \in S$ such that $\mathfrak{t}_1 \not\rightsquigarrow \mathfrak{t}_2$.

This ARS leads to the **Knuth-Bendix completion Algorithm** [D. Knuth, P. Bendix, Simple Words Problems in Universal Algebras, 1970].

The original Knuth-Bendix completion Algorithm works on sets of **identities** (unordered pairs of $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-terms) rather than elementary rewrite relations (ordered pairs of $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-terms).

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a reduction relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\lfloor \mathrm{C}{\cdot}\mathsf{V} \rfloor$.

Given a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, S)$, the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is used by computing a **normal form** of the set S and, when this normal form is a set S', by considering the TRS $(\mathcal{S}, \mathrm{C}{\cdot}\mathsf{V}, S')$ as the result. This TRS is a *completion* of $\mathcal{T}$.

Example

Let $\mathcal{S}$ be the signature containing two binary constants a and b. Let us consider the set of variables $\mathbb{N}$. Let $\rightsquigarrow$ be the $\succ, \omega$-Knuth-Bendix relation where $\succ$ satisfies $b \succ a$, and ω is the $\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}, \succ$-weight function defined by $\omega_{\mathrm{C}\cdot\mathbb{N}} := 1$, $\omega{\cdot}a = 3$, and $\omega{\cdot}b = 2$.

Let $S := \{(a \lfloor a12 \rfloor 3, a1 \lfloor b23 \rfloor)\}$. In $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$, we have

$$S \Rightarrow \{(a \lfloor a12 \rfloor 3, a1 \lfloor b23 \rfloor), (\mathfrak{t}_1, \mathfrak{t}_2)\}$$

with the following definitions.

By denoting by r the element of S, we have that $(r, 1, r)$ is a critical data of the TRS $(\mathcal{S}, \mathsf{V}, S)$. The critical term associated with this critical data is $\mathfrak{t} := a \lfloor a \lfloor a1^{(1)}2^{(1)} \rfloor 3^{(1)} \rfloor 3$. The associated critical pair is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := a \lfloor a1^{(1)}2^{(1)} \rfloor \lfloor b3^{(1)}3 \rfloor$ and $\mathfrak{s}_2 := a \lfloor a1^{(1)} \lfloor b2^{(1)}3^{(1)} \rfloor \rfloor 3$.

We have moreover

$$\mathfrak{s}_1 \Rightarrow a1^{(1)} \lfloor b2^{(1)} \lfloor b3^{(1)}3 \rfloor \rfloor =: \mathfrak{t}_2 \qquad \text{and} \qquad \mathfrak{s}_2 \Rightarrow a1^{(1)} \lfloor b \lfloor b2^{(1)}3^{(1)} \rfloor 3 \rfloor =: \mathfrak{t}_1.$$

These two normal forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ satisfy $\mathfrak{t}_1 \rightsquigarrow \mathfrak{t}_2$.

Due to the following results, the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$, where $\mathcal{S}$ is a signature, V is a set of variables, and $\rightsquigarrow$ is a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathsf{V}\rfloor$, computes exactly what it is designed to compute.

Exercise ○○○○○

Show that $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is neither terminating nor confluent, where $\mathcal{S}$ is a signature, V is a set of variables, and $\rightsquigarrow$ is a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathsf{V}\rfloor$.

Theorem [Normal forms of $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$]

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS and $\rightsquigarrow$ be a $\mathcal{T}$-compatible reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathsf{V}\rfloor$. All normal forms of the set $\rightarrow$ in $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ which are different of Fail are binary relations S on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathsf{V}\rfloor$ such that $\mathcal{T}' := (\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ is a convergent TRS such that $\equiv_{\mathcal{T}'} = \equiv_{\mathrm{C}\cdot\mathcal{T}}$.

In other terms, given a **terminating** TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$, a normal form of of $\rightarrow$ in $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is an elementary rewrite relation S such that the TRSs $(\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ and $\mathrm{C}\cdot\mathcal{T}$ are **convertibility equivalent**, and $(\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ is **convergent** (even if $\mathrm{C}\cdot\mathcal{T}$ is not).

Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m, and $\to$ is defined through the rewrite rule $r_1 := m \lfloor m12 \rfloor \lfloor m23 \rfloor \to 2$. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical data of $\mathcal{T}$ are $(r_1, 1, r_1)$ and $(r_1, 2, r_1)$.

The critical term associated with the first critical data is

$$\mathfrak{t} := m \lfloor m \lfloor m1^{(1)}2^{(1)} \rfloor \lfloor m2^{(1)}3^{(1)} \rfloor \rfloor \lfloor m \lfloor m2^{(1)}3^{(1)} \rfloor 3 \rfloor$$

and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where

$$\mathfrak{s}_1 := m2^{(1)}3^{(1)} \quad \text{and} \quad \mathfrak{s}_2 := m2^{(1)} \lfloor m \lfloor m2^{(1)}3^{(1)} \rfloor 3 \rfloor.$$

These two $\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}$-terms are normal forms of $\mathrm{C}{\cdot}\mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_2 := (\mathfrak{s}_2, \mathfrak{s}_1)$, in this order, since $\mathfrak{s}_2$ contains $\mathfrak{s}_1$ as a subterm.

Similarly, the second critical data makes $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ to add to the set $\{r_1, r_2\}$ the new rewrite rule

$$r_3 := \left(m \lfloor m1 \lfloor m1^{(1)}2^{(1)} \rfloor \rfloor 2^{(1)}, m1^{(1)}2^{(1)}\right).$$

We can check (with some further work) that $\{r_1, r_2, r_3\}$ is a normal form of $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$. Therefore, the TRS $(\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}, \{r_1, r_2, r_3\})$ is convergent.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m, and $\to$ is defined through the rewrite rules $r_1 := m\underline{m12}3 \to m1\underline{m23}$ and $r_2 := m11 \to 1$. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical term associated with the critical data $(r_1, 1, r_1)$ of $\mathcal{T}$ is $m\underline{m1^{(1)}1^{(1)}}3$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m1^{(1)}\underline{m1^{(1)}3}$ and $\mathfrak{s}_2 := m1^{(1)}3$. These two $\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}$-terms are normal forms of $\mathrm{C}{\cdot}\mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_3 := (\mathfrak{s}_1, \mathfrak{s}_2)$.

Let the TRS $\mathcal{T}' := (\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}, \{r_1, r_2, r_3\})$. The critical term associated with the critical data $(r_1, 1, r_3)$ of $\mathcal{T}'$ is $m\underline{m1^{(2)}\underline{m1^{(2)}3^{(1)}}}3$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m1^{(2)}\underline{m\underline{m1^{(2)}3^{(1)}}3}$ and $\mathfrak{s}_2 := m\underline{m1^{(2)}3^{(1)}}3$. We have that $\mathfrak{s}_1' := m1^{(3)}\underline{m1^{(3)}\underline{m3^{(1)}3}}$ and $\mathfrak{s}_2' := m1^{(3)}\underline{m3^{(1)}3}$ are respective normal forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathcal{T}'$. Therefore, $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ adds to $\{r_1, r_2, r_3\}$ the new rewrite rule $r_4 := (\mathfrak{s}_1', \mathfrak{s}_2')$.

Exercise ●●●○○

Let us consider the previous TRS $\mathcal{T}$. Show that for any $\mathcal{T}$-compatible reduction relation $\rightsquigarrow$, the set $\to$ admits no normal different from Fail in the ARS $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$. In other words, prove that there is no confluent completion of $\mathcal{T}$.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m and $\to$ is defined through the rewrite rule $r_1 := m\,\underline{m\,\underline{12}\,3}\,4 \to m1\,\underline{m2\,\underline{m34}}$ [C. Chenavier, C. Cordero, S. Giraudo, Quotients of the magmatic operad: lattice structures and convergent rewrite systems, 2019]. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical term associated with the critical data $(r_1, 1, r_1)$ of $\mathcal{T}$ is $m\,\underline{m\,\underline{m\,\underline{m1^{(1)}2^{(1)}}\,3^{(1)}}\,4^{(1)}}\,4$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m\,\underline{m1^{(1)}2^{(1)}}\,\underline{m3^{(1)}\,\underline{m4^{(1)}4}}$ and $\mathfrak{s}_2 := m\,\underline{m1^{(1)}\,\underline{m2^{(1)}\,\underline{m3^{(1)}4^{(1)}}}}\,4$. These two $\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}$-terms are normal forms of $\mathrm{C}{\cdot}\mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_2 := (\mathfrak{s}_1, \mathfrak{s}_2)$.

Let the TRS $\mathcal{T}' := (\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}, \{r_1, r_2\})$. The critical term associated with the critical data $(r_1, 11, r_1)$ of $\mathcal{T}'$ is $m\,\underline{m\,\underline{m\,\underline{m\,\underline{m1^{(1)}2^{(1)}}\,3^{(1)}}\,4^{(1)}}\,3}\,4$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m\,\underline{m\,\underline{m1^{(1)}2^{(1)}}\,3^{(1)}}\,\underline{m4^{(1)}\,\underline{m23}}$ and $\mathfrak{s}_2 := m\,\underline{m\,\underline{m1^{(1)}\,\underline{m2^{(1)}\,\underline{m3^{(1)}4^{(1)}}}}\,3}\,4$. We have that $\mathfrak{s}'_1 := m1^{(1)}\,\underline{m2^{(1)}\,\underline{m3^{(1)}\,\underline{m4^{(1)}\,\underline{m34}}}}$ and, by using the rewrite rule r_2, $\mathfrak{s}'_2 := m1^{(1)}\,\underline{m2^{(1)}\,\underline{m\,\underline{m3^{(1)}\,\underline{m4^{(1)}3}}\,4}}$ are respective normal forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathcal{T}'$. Therefore, $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$ adds to $\{r_1, r_2\}$ the new rewrite rule $r_3 := (\mathfrak{s}'_1, \mathfrak{s}'_2)$.

Exercise ●●●○○

Let us consider the previous TRS $\mathcal{T}$. Show that $\mathcal{T}$ leads to a convergent TRS formed by eleven rewrite rules.

9. Universal algebra

9.1. Varieties of algebras

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

An $\mathcal{S}$-algebra $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ is an *$\mathcal{S}$-subalgebra* of $\mathcal{A}$ if $X' \subseteq X$ and for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any $x'_1, \dots, x'_n \in X'$,

$$\mathrm{op}' \cdot c \cdot x'_1 \cdot \dots \cdot x'_n = \mathrm{op} \cdot c \cdot x'_1 \cdot \dots \cdot x'_n.$$

Examples

Let MagC be the signature containing one nullary constant c and one binary constant m.

- The triple $\mathcal{A} := (\mathcal{P} \cdot \mathbb{Z}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \mathbb{N}$ and $\mathrm{op} \cdot m \cdot S_1 \cdot S_2 := S_1 \cap S_2$ is a MagC-algebra.
  The triple $(\mathcal{P} \cdot \mathbb{N}, \mathrm{MagC}, \mathrm{op})$ where op is the same function op as before, but restricted on $\mathcal{P} \cdot \mathbb{N}$, is a MagC-subalgebra of $\mathcal{A}$.
- The triple $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 0$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$ is a MagC-algebra.
  The triple $(\{2n : n \in \mathbb{N}\}, \mathrm{MagC}, \mathrm{op})$ where op is the same function op as before, but restricted on even natural numbers, is a MagC-subalgebra of $\mathcal{A}$.
- By considering the previous MagC-algebra $\mathcal{A}$, the triple $([\![k]\!], \mathrm{MagC}, \mathrm{op})$, $k \in \mathbb{N}$, where op is the same function op as before, but restricted on natural numbers smaller than or equal to k, is **not** a MagC-subalgebra of $\mathcal{A}$ because $\mathrm{op} \cdot m$ is not well-defined.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

Given $Y \subseteq X$, the *$\mathcal{S}$-subalgebra of $\mathcal{A}$ generated by Y* is the $\mathcal{S}$-subalgebra $\mathcal{A}^{(Y)}$ of $\mathcal{A}$ whose underlying set contains Y and is minimal w.r.t. inclusion among the underlying sets of all $\mathcal{S}$-subalgebras of $\mathcal{A}$ which satisfy this property.

When $\mathcal{A} = \mathcal{A}^{(Y)}$, Y is a *generating set* of $\mathcal{A}$. When for any $Y' \subseteq Y$, $\mathcal{A}^{(Y')} = \mathcal{A}^{(Y)}$ implies $Y' = Y$, Y is a *minimal generating set* of $\mathcal{A}$.

Examples

- Let the MagC-algebra $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 0$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$.
  We have $\mathcal{A}^{(\{1\})} = \mathcal{A}$. This follows by induction on $\mathbb{N}$. First, we have $0 \in \mathcal{A}^{(\{1\})}$ because $0 = \mathrm{op} \cdot c$. We have also $1 \in \mathcal{A}^{(\{1\})}$ trivially because 1 is an element of the generating set. Moreover, for any $n \in \mathbb{N} \setminus \{0\}$, $n = \mathrm{op} \cdot m \cdot \llcorner n - 1 \lrcorner \cdot 1$, so that, by induction hypothesis, $n \in \mathcal{A}^{(\{1\})}$.
- Let the MagC-algebra $\mathcal{A} := (\mathbb{Z}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 1$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 \times n_2$.
  The MagC-subalgebra $\mathcal{A}^{(\{2,3\})}$ of $\mathcal{A}$ has as underlying set $\left\{2^{k_1} 3^{k_2} : k_1, k_2 \in \mathbb{N}\right\}$.

Let I be a set and $(\mathcal{A}_i)_{i \in I}$ be a family such that for any $i \in I$, $\mathcal{A}_i := (X_i, \mathcal{S}, \mathrm{op}_i)$ is an $\mathcal{S}$-algebra.

The *product* of $(\mathcal{A}_i)_{i \in I}$ is the $\mathcal{S}$-algebra $\prod_{i \in I} \mathcal{A}_i := (X, \mathcal{S}, \mathrm{op})$ such that $X := \prod_{i \in I} X_i$ and for any $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, and any $\left(x_i^{(1)}\right)_{i \in I}, \dots, \left(x_i^{(n)}\right)_{i \in I} \in X$,

$$\mathrm{op}{\cdot}c{\cdot}\left(x_i^{(1)}\right)_{i \in I}{\cdot} \cdots {\cdot}\left(x_i^{(n)}\right)_{i \in I} := \left(\mathrm{op}_i{\cdot}c{\cdot}x_i^{(1)}{\cdot} \cdots {\cdot}x_i^{(n)}\right)_{i \in I}.$$

Example

Let the MagC-algebras $\mathcal{A}_1 := (\mathbb{N}, \mathrm{MagC}, \mathrm{op}_1)$, where $\mathrm{op}_1{\cdot}c := 0$ and $\mathrm{op}_1{\cdot}m{\cdot}n_1{\cdot}n_2 := n_1 + n_2$, and $\mathcal{A}_2 := (\mathbb{Z}, \mathrm{MagC}, \mathrm{op}_2)$, where $\mathrm{op}_2{\cdot}c := 1$ and $\mathrm{op}_2{\cdot}m{\cdot}n_1{\cdot}n_2 := n_1 \times n_2$.

By setting $\mathcal{A} := \prod_{i \in [2]} \mathcal{A}_i = (X, \mathcal{S}, \mathrm{op})$, we have

- $X = \mathbb{N} \times \mathbb{Z}$;
- $\mathrm{op}{\cdot}c = (0, 1)$;
- For any $\left(n_1^{(1)}, n_2^{(1)}\right), \left(n_1^{(2)}, n_2^{(2)}\right) \in X$, $\mathrm{op}{\cdot}m{\cdot}\left(n_1^{(1)}, n_2^{(1)}\right){\cdot}\left(n_1^{(2)}, n_2^{(2)}\right) = \left(n_1^{(1)} + n_1^{(2)}, n_2^{(1)} \times n_2^{(2)}\right)$.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ and $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ be two $\mathcal{S}$-algebras.

An *$\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$* is a function $\phi : X \to X'$ such that for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any $x_1, \dots, x_n \in X$,

$$\phi \cdot \llcorner \mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_n \lrcorner = \mathrm{op}' \cdot c \cdot \llcorner \phi \cdot x_1 \lrcorner \cdot \cdots \cdot \llcorner \phi \cdot x_n \lrcorner .$$

Example

Let the MagC-algebras $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \epsilon$ and $\mathrm{op} \cdot m \cdot w_1 \cdot w_2 := w_1 \centerdot w_2$, and $\mathcal{A}' := (\mathbb{N}, \mathrm{MagC}, \mathrm{op}')$, where $\mathrm{op}' \cdot c := 0$ and $\mathrm{op}' \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$.

The function ℓ, which sends any $w \in \{a, b\}^*$ to the length of w, is an MagC-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$. Indeed,

$$\ell \cdot \llcorner \mathrm{op} \cdot c \lrcorner = \ell \cdot \epsilon = 0 = \mathrm{op}' \cdot c$$

and, for any $w_1, w_2 \in \{a, b\}^*$,

$$\ell \cdot \llcorner \mathrm{op} \cdot m \cdot w_1 \cdot w_2 \lrcorner = \ell \cdot \llcorner w_1 \centerdot w_2 \lrcorner = \ell \cdot w_1 + \ell \cdot w_2 = \mathrm{op}' \cdot m \cdot \llcorner \ell \cdot w_1 \lrcorner \cdot \llcorner \ell \cdot w_2 \lrcorner .$$

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra.

An equivalence relation $\equiv$ on X is a *congruence on* $\mathcal{A}$ if for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, any $x_1, \dots, x_i, x'_i, \dots x_n \in X$, $i \in [n]$, $x_i \equiv x'_i$ implies

$$\mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_i \cdots \cdot x_n \equiv \mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x'_i \cdots \cdot x_n.$$

The **$\equiv$-equivalence class** of $x \in X$ in $\mathcal{A}$ is denoted by $[x]_{\equiv}$.

The *quotient* of $\mathcal{A}$ by a congruence $\equiv$ on $\mathcal{A}$ is the $\mathcal{S}$-algebra $\mathcal{A}/\equiv := (X/\equiv, \mathcal{S}, \mathrm{op}/\equiv)$ such that $\mathrm{op}/\equiv$ is defined, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$, by

$$\mathrm{op}/\equiv \cdot c \cdot [x_1]_{\equiv} \cdot \cdots \cdot [x_n]_{\equiv} := [\mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_n]_{\equiv}.$$

Example

Let the MagC-algebra $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := \epsilon$ and $\mathrm{op} \cdot m \cdot w_1 \cdot w_2 := w_1 . w_2$.
Let $\equiv$ be the equivalence relation on $\{a, b\}^*$ such that, for any $w_1, w_2 \in \{a, b\}^*$, $w_1 \equiv w_2$ if $\ell_a \cdot w_1 = \ell_a \cdot w_2$ and $\ell_b \cdot w_1 = \ell_b \cdot w_2$. It is easy to check that $\equiv$ is a congruence of $\mathcal{A}$.
The underlying set $\{a, b\}^*/\equiv$ of $\mathcal{A}/\equiv$ admits $\{a^{n_a} b^{n_b} : n_a, n_b \in \mathbb{N}\}$ as set of representatives.
Moreover, $\mathrm{op}/\equiv \cdot c = [\epsilon]_{\equiv}$ and, for any $n_a, n_b, n'_a, n'_b \in \mathbb{N}$,

$$\mathrm{op}/\equiv \cdot m \cdot [a^{n_a} b^{n_b}]_{\equiv} \cdot \left[a^{n'_a} b^{n'_b}\right]_{\equiv} = \left[a^{n_a + n'_a} b^{n_b + n'_b}\right]_{\equiv}.$$

Let X and X' be two sets and $\phi : X \to X'$ be a function.

- The *kernel* of ϕ is the equivalence relation $\mathrm{Ker}\cdot\phi$ on X defined by $(x_1, x_2) \in \mathrm{Ker}\cdot\phi$ if $\phi\cdot x_1 = \phi\cdot x_2$.
- The *image* of ϕ is the set $\mathrm{Im}\cdot\phi := \{x' \in X' : \text{ there exists } x \in X \text{ such that } \phi\cdot x = x'\}$.

Theorem [First Isomorphism Theorem]

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, $\mathcal{A}' := (X', \mathcal{S}, \mathrm{op}')$ be two $\mathcal{S}$-algebras, and ϕ be an $\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$.

1. The equivalence relation $\mathrm{Ker}\cdot\phi$ is a congruence on $\mathcal{A}$.
2. The triple $\mathcal{A}'' := (\mathrm{Im}\cdot\phi, \mathcal{S}, \mathrm{op}'')$ is an $\mathcal{S}$-subalgebra of $\mathcal{A}'$, where, for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, $\mathrm{op}''\cdot c$ is the restriction of $\mathrm{op}'\cdot c$ as an n-operation on $\mathrm{Im}\cdot\phi$.
3. The function $\bar{\phi} : X/\mathrm{Ker}\cdot\phi \to \mathrm{Im}\cdot\phi$ defined for any $x \in X$ by $\bar{\phi}\cdot[x]_{\mathrm{Ker}\cdot\phi} := \phi\cdot x$ is an $\mathcal{S}$-algebra isomorphism between $\mathcal{A}/\mathrm{Ker}\cdot\phi$ and $\mathcal{A}''$.

Exercise ●●●○○

Prove Theorem [First Isomorphism Theorem].

Let $\mathcal{S}$ be a signature and $\mathfrak{C}$ be a **class** of $\mathcal{S}$-algebras.

The class $\mathfrak{C}$ is

- *closed under homomorphic images* when for any $\mathcal{A} \in \mathfrak{C}$ and any $\mathcal{S}$-algebra $\mathcal{A}'$, if there is a surjective $\mathcal{S}$-algebra morphism from $\mathcal{A}$ to $\mathcal{A}'$, then $\mathcal{A}' \in \mathfrak{C}$;
- *closed under subalgebras* when for any $\mathcal{A} \in \mathfrak{C}$, any $\mathcal{S}$-subalgebra of $\mathcal{A}$ belongs to $\mathfrak{C}$;
- *closed under products* when for any set I and any family $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{S}$-algebras belonging to $\mathfrak{C}$, $\prod_{i \in I} \mathcal{A}_i \in \mathfrak{C}$.

Note that by Theorem [First Isomorphism Theorem], if $\mathfrak{C}$ is closed under homomorphic images, then for any $\mathcal{A} \in \mathfrak{C}$ and any congruence $\equiv$ on $\mathcal{A}$, the quotient $\mathcal{A}/\equiv$ belongs to $\mathfrak{C}$.

Note also that the closure under homomorphic images implies that if $\mathcal{A} \in \mathfrak{C}$, then any $\mathcal{S}$-algebra isomorphic to $\mathcal{A}$ belongs to $\mathfrak{C}$.

Definition

Let $\mathcal{S}$ be a signature. A *variety* of $\mathcal{S}$-algebras is a nonempty class $\mathfrak{C}$ of $\mathcal{S}$-algebras which is closed under homomorphic images, subalgebras, and products.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The $\mathcal{S}, \mathsf{V}$-*term algebra* is the $\mathcal{S}$-algebra $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} := (\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \mathcal{S}, \mathrm{op})$ such that op is defined, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$, by

$$\mathrm{op} \cdot c \cdot \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_n := c \, \mathfrak{t}_1 \dots \mathfrak{t}_n.$$

The $\mathcal{S}, \emptyset$-term algebra is the *ground $\mathcal{S}$-term algebra*. Note that if $\mathcal{S} \cdot 0 = \emptyset$, then the underlying set of $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ is empty.

Example

The $\mathrm{MagC}, \mathbb{N}$-term algebra $\mathbf{T} \cdot \mathrm{MagC} \cdot \mathbb{N}$ admits the set of $\mathrm{MagC}, \mathbb{N}$-terms. Moreover, we have for instance

$$\mathrm{op} \cdot c = c$$

and

$$\mathrm{op} \cdot m \cdot m1 \llcorner mc2 \lrcorner \cdot m23 = m \llcorner m1 \llcorner mc2 \lrcorner \lrcorner \llcorner m23 \lrcorner.$$

Besides, the underlying set of the ground $\mathcal{S}$-term algebra is

$$X := \{c, mcc, m \llcorner mcc \lrcorner c, mc \llcorner mcc \lrcorner, \dots\}$$

and the graded set (X, ℓ_c) is a combinatorial graded set and its integer sequence is the one of Catalan numbers.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\iota : \mathsf{V} \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ be the function such that for any $v \in \mathsf{V}$, $\iota \cdot v$ is the leaf decorated by v.

Theorem [Free $\mathcal{S}$-algebras]

For any signature $\mathcal{S}$, any set of variables V, any $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, and any V, X-assignment α, there exists a unique $\mathcal{S}$-algebra morphism ϕ from $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V}$ to $\mathcal{A}$ such that $\alpha = \phi \circ \iota$.

The class of $\mathcal{S}$-algebras together with $\mathcal{S}$-algebra morphisms forms a **category**.

Theorem [Free $\mathcal{S}$-algebras] says that $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V}$ is a **free object** in this category.

For any set X, the *empty X-assignment* is the $\emptyset, X$-assignment $\emptyset$ having an empty domain.

For any $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, $\mathrm{ev}_{\mathcal{A}, \emptyset}$ is the **unique $\mathcal{S}$-algebra morphism** from $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ to $\mathcal{A}$. Therefore, $\mathbf{T} \cdot \mathcal{S} \cdot \emptyset$ is an **initial object** in the category of $\mathcal{S}$-algebras.

9.2. Equational presentations

Definition

An $\mathcal{S}, \mathsf{V}$-*equational presentation* is a triple $(\mathcal{S}, \mathsf{V}, \sim)$ where

- $\mathcal{S}$ is a signature, called the *underlying signature*;
- V is a set of variables, called the *underlying set of variables*;
- $\sim$ is a binary relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$, called the *elementary identity relation*.

Remark that this definition looks like the one of TRSs but there is no condition on $\sim$ here.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation. If $\mathfrak{t}$ and $\mathfrak{t}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t} \sim \mathfrak{t}'$, then the pair $(\mathfrak{t}, \mathfrak{t}')$ is an *elementary identity* of $\mathcal{E}$.

Examples

- Let $\mathbf{Monoids} := (\mathrm{MagC}, \mathbb{N}, \sim)$ be the $\mathrm{MagC}, \mathbb{N}$-equational presentation such that $\sim$ is defined by $m \llcorner m12 \lrcorner 3 \sim m1 \llcorner m23 \lrcorner$, $m1c \sim 1$, and $mc1 \sim 1$.
- Let $\mathbf{BSLattices} := (\mathrm{MagC}, \mathbb{N}, \sim)$ be the $\mathrm{MagC}, \mathbb{N}$-equational presentation such that $\sim$ is defined by $m \llcorner m12 \lrcorner 3 \sim m1 \llcorner m23 \lrcorner$, $m1c \sim 1$, $mc1 \sim 1$, $m12 \sim m21$, and $m11 \sim 1$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

Let $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra. Two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *$\mathcal{A}$-equivalent* if for any V, X-assignment α, $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'$.

An *algebra over* $\mathcal{E}$ is an $\mathcal{S}$-algebra $\mathcal{A}$ such that for any elementary identity $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{E}$, $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{A}$-equivalent.

Examples

- Let the MagC-algebra $\mathcal{A} := (\mathbb{N}, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op} \cdot c := 0$ and $\mathrm{op} \cdot m \cdot n_1 \cdot n_2 := n_1 + n_2$. The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m 1 \llcorner m 1 2 \lrcorner$ and $\mathfrak{t}' := m \llcorner m 2 1 \lrcorner 1$ are $\mathcal{A}$-equivalent. Indeed, for any $\mathbb{N}, \mathbb{N}$ assignment α, by setting $n_1 := \alpha \cdot 1$ and $n_2 := \alpha \cdot 2$,
$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} = n_1 + (n_1 + n_2) = 2 n_1 + n_2 = (n_2 + n_1) + n_1 = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'.$$
- The class of algebras over **Monoids** is the class of monoids.
- The class of algebras over **BSLattices** is the class of bounded semilattices.

Here is a list of some important varieties appearing frequently in algebraic combinatorics.

- The variety of monoids, described by the equational presentation **Monoids**.
- The variety of bounded semilattices, described by the equational presentation **BSLattices**.
- The variety of groups, described by the equational presentation $\mathbf{Groups} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \lfloor m12 \rfloor 3 \sim m1 \lfloor m23 \rfloor$, $me1 \sim 1$, $m1e \sim 1$, $m \lfloor i1 \rfloor 1 \sim e$, and $m1 \lfloor i1 \rfloor \sim e$.
- The variety of idempotent semigroups (also called *bands*), described by the equational presentation $\mathbf{Bands} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains one binary constant m and $\sim$ is defined by $m \lfloor m12 \rfloor 3 \sim m1 \lfloor m23 \rfloor$ and $m11 \sim 1$.
- The variety of *duplicial algebras* [C. Brouder, A. Frabetti, QED Hopf algebras on planar binary trees, 2003], described by the equational presentation $\mathbf{DuplicialAlgebras} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains two binary constant $\ll$ and $\gg$, and $\sim$ is defined by $\ll \lfloor \ll 12 \rfloor 3 \sim \ll 1 \lfloor \ll 23 \rfloor$, $\gg \lfloor \gg 12 \rfloor 3 \sim \gg 1 \lfloor \gg 23 \rfloor$, and $\gg \lfloor \ll 12 \rfloor 3 \sim \ll 1 \lfloor \gg 23 \rfloor$.
- The variety of *nonassociative permutative algebras* [M. Livernet, A rigidity theorem for pre-Lie algebras, 2006], described by the equational presentation $\mathbf{NAPAlgebras} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ contains one binary constant g, and $\sim$ is defined by $\mathsf{g} \lfloor \mathsf{g}12 \rfloor 3 \sim \mathsf{g} \lfloor \mathsf{g}13 \rfloor 2$.

Theorem [Birkhoff's Variety Theorem]

Let $\mathcal{S}$ be a signature and V be an **infinite** set of variables. A **nonempty** class $\mathfrak{C}$ of $\mathcal{S}$-algebras is a **variety** iff there exists an $\mathcal{S}, \mathsf{V}$**-equational presentation** $\mathcal{E}$ such that $\mathfrak{C}$ is the class of algebras over $\mathcal{E}$.

This result comes from [G. Birkhoff, On the Structure of Abstract Algebras, 1935] and is also known as the **Birkhoff HSP Theorem**.

When $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is an $\mathcal{S}, \mathsf{V}$-equational presentation with V infinite, by [Birkhoff's Variety Theorem], the class of algebras over $\mathcal{E}$ is a variety $\mathcal{V}$. We call $\mathcal{V}$ the *variety of $\mathcal{E}$*.

Examples

- Since the class $\mathfrak{C}$ of monoids is closed under homomorphic images, subalgebras, and products, $\mathfrak{C}$ is a variety. By Theorem [Birkhoff's Variety Theorem], there exists an $\mathcal{S}, \mathsf{V}$-equational presentation of $\mathfrak{C}$. This is the $\mathrm{MagC}, \mathbb{N}$-equational presentation $\mathbf{Monoids}$.
- By Theorem [Birkhoff's Variety Theorem], $\mathbf{BSLattices}$ is an $\mathrm{MagC}, \mathbb{N}$-equational presentation of a variety. This is the variety of bounded semilattices.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

Two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{E}$-*semantically equivalent* if for any algebra $\mathcal{A}$ over $\mathcal{E}$, $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{A}$-equivalent. This property is denoted by $\mathfrak{t} \approx_{\mathcal{E}} \mathfrak{t}'$.

Examples

- The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m1\llcorner m12 \lrcorner$ and $\mathfrak{t}' := m\llcorner m21\lrcorner 1$ are not **Monoids**-semantically equivalent. Indeed, consider the MagC-algebra $\mathcal{A} := (\{a, b\}^*, \mathrm{MagC}, \mathrm{op})$, where $\mathrm{op}{\cdot}c := \epsilon$ and $\mathrm{op}{\cdot}m{\cdot}w_1{\cdot}w_2 := w_1 \centerdot w_2$ is such that, by setting α as an $\mathbb{N}, \{a, b\}^*$-assignment satisfying $\alpha{\cdot}1 = a$ and $\alpha{\cdot}2 = b$, $\mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\mathfrak{t} = aab \neq baa = \mathrm{ev}_{\mathcal{A},\alpha}{\cdot}\mathfrak{t}'$.
- The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m\llcorner m12\lrcorner\llcorner m21\lrcorner$ and $\mathfrak{t}' := m12$ are **BSLattices**-semantically equivalent.
- The $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := mc1$ and $\mathfrak{t}' := c$ are not **BSLattices**-semantically equivalent.

Exercise ●○○○○

Prove the last two properties of the previous examples.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an $\mathcal{S}, \mathsf{V}$-equational presentation.

When $\sim$ is an elementary rewrite relation, $\mathcal{E}$ is *TRS-like*. In this case, $\mathcal{E}$ is a TRS.

Examples

- The equational presentations **Monoids**, **BSLattices**, and **Groups** are TRS-like.
- The equational presentation $(\mathrm{MagC}, \mathbb{N}, \sim)$ such that $\sim$ is defined by $1 \sim m11$ is not TRS-like.
- The equational presentation $(\mathrm{MagC}, \mathbb{N}, \sim)$ such that $\sim$ is defined by $mcc \sim m1c$ is not TRS-like.

When $\mathcal{E}$ is TRS-like, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *$\mathcal{E}$-syntactically equivalent* if $\mathfrak{t} \equiv \mathfrak{t}'$ where $\equiv$ is the **convertibility relation** of $\mathcal{E}$.

Example

Let the $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t} := m \lfloor mc1 \rfloor \lfloor m3 \lfloor m \lfloor m3c \rfloor 2 \rfloor \rfloor$ and $\mathfrak{t}' := m \lfloor m12 \rfloor 3$. In **BSLattices**, we have

$$\mathfrak{t}' \Rightarrow m1 \lfloor m23 \rfloor \Leftarrow m \lfloor mc1 \rfloor \lfloor m23 \rfloor \Rightarrow m \lfloor mc1 \rfloor \lfloor m32 \rfloor \Leftarrow m \lfloor mc1 \rfloor \lfloor m \lfloor m33 \rfloor 2 \rfloor \Leftarrow m \lfloor mc1 \rfloor \lfloor m \lfloor m3 \lfloor m3c \rfloor \rfloor 2 \rfloor \Rightarrow \mathfrak{t}$$

so that $\mathfrak{t}$ and $\mathfrak{t}'$ are **BSLattices**-syntactically equivalent.

Theorem [Equivalence of the $\mathcal{E}$-semantic and $\mathcal{E}$-syntactic relations]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a TRS-like $\mathcal{S}, \mathsf{V}$-equational presentation. The $\mathcal{E}$-semantic equivalence relation and the $\mathcal{E}$-syntactic equivalence relation coincide.

This result is known as the **Birkhoff Theorem on identities** [G. Birkhoff, On the Structure of Abstract Algebras, 1935].

Due to this property, given a TRS-like equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ *$\mathcal{E}$-equivalent* if $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{E}$-semantically equivalent, or, equivalently, $\mathcal{E}$-syntactically equivalent. This property is denoted by $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$.

Exercise ●●○○○

Let the $\mathcal{S}, \mathbb{N}$-equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N}, \sim)$ such that $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant f, and $\sim$ is defined by $f1\llcorner f23 \lrcorner \sim f\llcorner f12 \lrcorner 3$, $fe1 \sim 1$, and $f1\llcorner i1 \lrcorner \sim e$.

Show that the $\mathcal{S}, \mathbb{N}$-terms $f1e$ and 1 are not $\mathcal{E}$-equivalent.

Proposition [$\mathcal{S}$-equivalence and quotients of free $\mathcal{S}, \mathsf{V}$-term algebras]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a equational presentation. The $\mathcal{E}$-equivalence relation $\approx_{\mathcal{E}}$ is a congruence on $\mathbf{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be an equational presentation. Let $\bar{\iota} : \mathsf{V} \to \lfloor \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V} \rfloor / \approx_{\mathcal{E}}$ be the function such that for any $v \in \mathsf{V}$, $\bar{\iota}{\cdot}v := [\iota{\cdot}v]_{\approx_{\mathcal{E}}}$.

Theorem [Free algebras over equational presentations]

For any equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$, any algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ over $\mathcal{E}$, and any V, X-assignment α, there exists a unique $\mathcal{S}$-algebra morphism ϕ from $\mathbf{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}/\approx_{\mathcal{E}}$ to $\mathcal{A}$ such that $\alpha = \phi \circ \bar{\iota}$.

The class of algebras over $\mathcal{E}$ together with $\mathcal{S}$-algebra morphisms forms a category.

Theorem [Free algebras over equational presentations] says that $\mathbf{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}/\approx_{\mathcal{E}}$ is a free object in this category.

For any algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ over $\mathcal{E}$, the function $[\mathfrak{t}]_{\approx_{\mathcal{E}}} \mapsto \mathrm{ev}_{\mathcal{A},\emptyset}{\cdot}\mathfrak{t}$ is the unique $\mathcal{S}$-algebra morphism from $\mathbf{T}{\cdot}\mathcal{S}{\cdot}\emptyset/\approx_{\mathcal{E}}$ to $\mathcal{A}$. Therefore, $\mathbf{T}{\cdot}\mathcal{S}{\cdot}\emptyset/\approx_{\mathcal{E}}$ is an initial object in the category of algebras over $\mathcal{E}$.

Universal algebra

9.3. Word problem

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *word problem* on an equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is the **decision problem** whose input is two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and whose question is whether $\mathfrak{t} \approx_{\mathcal{E}} \mathfrak{t}'$.

Theorem [Undecidability of the word problem]

There exist equational presentations $\mathcal{E}$ such that the word problem on $\mathcal{E}$ is undecidable.

Proof. Let the equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N}, \sim)$ where $\mathcal{S}$ is the signature containing two nullary constants K and S, and a binary constant a, and $\sim$ is defined by $\mathsf{a} \lfloor \mathsf{a} \mathsf{K} 1 \rfloor 2 \sim 1$ and $\mathsf{a} \lfloor \mathsf{a} \lfloor \mathsf{a} \mathsf{S} 1 \rfloor 2 \rfloor 3 \sim \mathsf{a} \lfloor \mathsf{a} 1 3 \rfloor \lfloor \mathsf{a} 2 3 \rfloor$. This equational presentation, coming from **combinatory logic**, is known to have an undecidable word problem.

Example

An instance of the word problem on **Groups** is formed by the $\mathcal{S}, \mathbb{N}$-terms $\mathfrak{t} := i \lfloor m 1 2 \rfloor$ and $\mathfrak{t}' := m \lfloor i 2 \rfloor \lfloor i 1 \rfloor$. We can check that the answer is yes since $\mathfrak{t} \approx_{\mathbf{Groups}} \mathfrak{t}'$.

Theorem [Convergence and decidability of words problems]

Let $\mathcal{E}$ be a TRS-like equational presentation. If $\mathcal{E}$ is convergent and its elementary rewrite relation is finite, then the word problem on $\mathcal{E}$ is decidable.

When $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is a convergent TRS-like equational presentation, an algorithm to decide for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ whether $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$ consists in computing the unique normal form of $\mathfrak{t}$ and of $\mathfrak{t}'$ of $\mathcal{E}$ and checking if they are equal.

Example

By considering the convergent TRS-like equational presentation **Monoids**, let $\mathfrak{t} := m\lfloor m1\lfloor mc2\rfloor\rfloor\lfloor m13\rfloor$ and $\mathfrak{t}' := m\lfloor m12\rfloor\lfloor m\lfloor m1c\rfloor\lfloor mc3\rfloor\rfloor$. By denoting by $\Rightarrow$ the rewrite relation of **Monoids**, we have

$$\mathfrak{t} \Rightarrow m\lfloor m12\rfloor\lfloor m13\rfloor \Rightarrow m1\lfloor m2\lfloor m13\rfloor\rfloor := \mathfrak{s}$$

and

$$\mathfrak{t}' \Rightarrow m\lfloor m12\rfloor\lfloor m\lfloor m1c\rfloor\lfloor mc3\rfloor\rfloor \Rightarrow m\lfloor m12\rfloor\lfloor m1\lfloor mc3\rfloor\rfloor \Rightarrow m\lfloor m12\rfloor\lfloor m13\rfloor \Rightarrow m1\lfloor m2\lfloor m13\rfloor\rfloor = \mathfrak{s}.$$

Since $\mathfrak{s}$ is a normal form of **Monoids** and $\mathfrak{s} \in \mathfrak{t} \Downarrow \mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathbf{Monoids}} \mathfrak{t}'$.

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a **convergent TRS-like** equational presentation.

The *algebra of normal forms of* $\mathcal{E}$ is the $\mathcal{S}$-algebra $\mathcal{A}_{\Rightarrow} \cdot \mathcal{E} := (X, \mathcal{S}, \mathrm{op})$ such that X is the set of normal forms of $\mathcal{E}$, and for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and any normal forms $\mathfrak{t}_1$, ..., $\mathfrak{t}_n$ of $\mathcal{E}$, $\mathrm{op} \cdot c \cdot \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_n$ is the unique normal form of the $\mathcal{S}, \mathsf{V}$-term $c\, \mathfrak{t}_1 \ldots \mathfrak{t}_n$ of $\mathcal{E}$.

Theorem [Algebras of normal forms and free algebras]

Let $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ be a convergent TRS-like equational presentation and X be the underlying set of $\mathcal{A}_{\Rightarrow} \cdot \mathcal{E}$. Let $\phi : \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} / \equiv_{\mathcal{E}} \to X$ be the function such that $\phi \cdot [\mathfrak{t}]_{\equiv_{\mathcal{E}}}$ is the unique normal form in the future of $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ in $\mathcal{E}$. Then, ϕ is an $\mathcal{S}$-algebra isomorphism from $\mathbf{T} \cdot \mathcal{S} \cdot \mathsf{V} / \equiv_{\mathcal{E}}$ to $\mathcal{A}_{\Rightarrow} \cdot \mathcal{E}$.

Since $\mathcal{E}$ is a convergent TRS, by Proposition [Church-Rosser property], the function ϕ of the statement of Theorem [Algebras of normal forms and free algebras] is well-defined.

Example

The MagC-algebra $\mathbf{T} \cdot \mathrm{MagC} \cdot \mathbb{N} / \equiv_{\mathbf{Monoids}}$ is isomorphic to the algebra $\mathcal{A}_{\Rightarrow} \cdot \mathbf{Monoids}$ of normal forms of $\mathbf{Monoids}$. The underlying set of $\mathcal{A}_{\Rightarrow} \cdot \mathbf{Monoids}$ is the set of $\mathrm{MagC}, \mathbb{N}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is a variable, or $\mathfrak{t} = c$, or $\mathfrak{t} = m i_1 \underbrace{m i_2 \underbrace{m i_3 \underbrace{\ldots \underbrace{m i_n i_{n+1}} \ldots}}}$ where $n \geqslant 1$ and for any $j \in [n+1]$, $i_j \in \mathbb{N}$.

Moreover, in $\mathcal{A}_{\Rightarrow} \cdot \mathbf{Monoids}$,

$$\mathrm{op} \cdot m \cdot \underbrace{m 1 \underbrace{m 2 3}} \cdot \underbrace{m 1 \underbrace{m 2 \underbrace{m 3 4}}} = m 1 \underbrace{m 2 \underbrace{m 3 \underbrace{m 1 \underbrace{m 2 \underbrace{m 3 4}}}}}.$$

10. Clones and varieties

10.1. Clones

Two varieties may be **equivalent** even if their underlying signatures are different.

Example

Let the equational presentation $\mathbf{PHeaps} := (\mathcal{S}', \mathbb{N}, \sim')$ where $\mathcal{S}'$ is the signature containing a nullary constant e' and a ternary constant p', and $\sim'$ is defined by $p'112 \sim' 2$, $p'122 \sim' 1$, and $p'\underbrace{p'123}45 \sim' p'12\underbrace{p'345}$.

The varieties of $\mathbf{Groups}$ and the variety of *pointed heaps* of $\mathbf{PHeaps}$ are equivalent.

Indeed, add to $\mathbf{PHeaps}$ a unary constant i' and a binary constant m', and set $m'12 \sim' p'1e'2$ and $i'1 \sim' p'e'1e'$. We have for instance

$$m'\underbrace{i'1}1 \approx_{\mathbf{PHeaps}} p'\underbrace{i'1}e'1 \approx_{\mathbf{PHeaps}} p'\underbrace{p'e'1e'}e'1 \approx_{\mathbf{PHeaps}} p'e'1\underbrace{p'e'e'1} \approx_{\mathbf{PHeaps}} p'e'11 \approx_{\mathbf{PHeaps}} e'.$$

Similarly, the five elementary identities of $\mathbf{Groups}$ hold in $\mathbf{PHeaps}$ on e', i', and m'.

Conversely, add to $\mathbf{Groups}$ a nullary constant e' and a ternary constant p', and set $e' \sim e$ and $p'123 \sim m\underbrace{m1\underbrace{i2}}3$. We have for instance

$$p'112 \quad \approx_{\mathbf{Groups}} \quad m\underbrace{m1\underbrace{i1}}2 \quad \approx_{\mathbf{Groups}} \quad me2 \quad \approx_{\mathbf{Groups}} \quad 2.$$

Similarly, the three elementary identities of $\mathbf{PHeaps}$ hold in $\mathbf{Groups}$ on e' and p'.

The variety of *Heaps* (pointed heaps are heaps with distinguished unit element) goes back at [H. Prüfer, Theorie der Abelschen Gruppen. I. Grundeigenschaften, 1924].

In order to explain such equivalences, we need algebraic structures yielding an **invariant** of an equational presentation, independent of the signature. **Abstract clones** are such invariants.

An *abstract clone* (or *clone* for short) is a triple $\left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ where

- $\mathcal{G}$ is a graded set, called the *underlying set*;
- for any $n, m \in \mathbb{N}$, $\gamma_{n,m}$ is a function of type

$$\mathcal{G}{\cdot}n \to \underbrace{\mathcal{G}{\cdot}m \to \cdots \to \mathcal{G}{\cdot}m}_{n \text{ times}} \to \mathcal{G}{\cdot}m,$$

  called the *n, m-superposition function*. In other words, $\gamma_{n,m}{\cdot}x$ is an n-operation on $\mathcal{G}{\cdot}m$ for any $x \in \mathcal{G}{\cdot}n$, $n \in \mathbb{N}$;
- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is an element of $\mathcal{G}{\cdot}n$, called the *i, n-projection*;
- the following identities hold:
  0.1. for any $n, m \in \mathbb{N}$, $i \in [n]$, and $x_1, \dots, x_n \in \mathcal{G}{\cdot}m$,

$$\gamma_{n,m}{\cdot}\mathbb{1}_{i,n}{\cdot}x_1{\cdot} \cdots {\cdot}x_n = x_i;$$

  0.2. for any $n \in \mathbb{N}$ and $x \in \mathcal{G}{\cdot}n$,

$$\gamma_{n,n}{\cdot}x{\cdot}\mathbb{1}_{1,n}{\cdot} \cdots {\cdot}\mathbb{1}_{n,n} = x;$$

  0.3. for any $n, m, k \in \mathbb{N}$, $x \in \mathcal{G}{\cdot}n$, $y_1, \dots, y_n \in \mathcal{G}{\cdot}m$, and $z_1, \dots, z_m \in \mathcal{G}{\cdot}k$,

$$\gamma_{m,k}{\cdot}\lfloor\gamma_{n,m}{\cdot}x{\cdot}y_1{\cdot} \cdots {\cdot}y_n\rfloor{\cdot}z_1{\cdot} \cdots z_m = \gamma_{n,k}{\cdot}x{\cdot}\lfloor\gamma_{m,k}{\cdot}y_1{\cdot}z_1{\cdot} \cdots {\cdot}z_m\rfloor{\cdot} \cdots {\cdot}\lfloor\gamma_{m,k}{\cdot}y_n{\cdot}z_1{\cdot} \cdots {\cdot}z_m\rfloor.$$

Example

Let the clone $\mathrm{SL} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- for any $n \in \mathbb{N}$, $\mathcal{G}{\cdot}n := \mathcal{P}{\cdot}[n] \setminus \emptyset$;
- for any $S \in \mathcal{G}{\cdot}n$, $n \in \mathbb{N}$, and $S_1, \dots, S_n \in \mathcal{G}{\cdot}m$, $m \in \mathbb{N}$,
$$\gamma_{n,m}{\cdot}S{\cdot}S_1{\cdot} \cdots {\cdot}S_n := \bigcup_{i \in S} S_i;$$
- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n} := \{i\}$.

We have for instance
$$\gamma_{5,9}{\cdot}\{1,3,4\}{\cdot}\{2\}{\cdot}\{6,8\}{\cdot}\{7,8\}{\cdot}\{2,3\}{\cdot}\{1,2,3,4\} = \{2\} \cup \{7,8\} \cup \{2,3\} = \{2,3,7,8\}.$$

The $2, 5$-projection is $\{2\}$.

There are in SL some nontrivial identities. For instance, for any $S_1, S_2 \in \mathcal{G}{\cdot}m$, $m \in \mathbb{N}$,
$$\gamma_{2,m}{\cdot}\{1,2\}{\cdot}S_1{\cdot}S_2 = S_1 \cup S_2 = S_2 \cup S_1 = \gamma_{2,m}{\cdot}\{1,2\}{\cdot}S_2{\cdot}S_1.$$

The *trivial clone* is the unique clone having its underlying set $\mathcal{G}$ satisfying that $\mathcal{G} \cdot n$ is a singleton for any $n \in \mathbb{N}$.

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ and $\mathcal{C}' := \left(\mathcal{G}', (\gamma'_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}'_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ be two clones.

A function $\phi : \mathcal{G} \to \mathcal{G}'$ is a *clone morphism* if

- for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $\phi \cdot x \in \mathcal{G}' \cdot n$;
- for any $n \in \mathbb{N}$ and $i \in [n]$, $\phi \cdot \mathbb{1}_{i,n} = \mathbb{1}'_{i,n}$;
- for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $y_1, \dots, y_n \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$,

$$\phi \cdot \llcorner \gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n \lrcorner = \gamma'_{n,m} \cdot \llcorner \phi \cdot x \lrcorner \cdot \llcorner \phi \cdot y_1 \lrcorner \cdot \cdots \cdot \llcorner \phi \cdot y_n \lrcorner.$$

The clone $\mathcal{C}'$ is a *subclone* of $\mathcal{C}$ if

- $\mathcal{G}'$ is a sub-graded set of $\mathcal{G}$;
- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}'_{i,n} = \mathbb{1}_{i,n}$;
- for any $n, m \in \mathbb{N}$, $x \in \mathcal{G}' \cdot n$, and $y_1, \dots, y_n \in \mathcal{G}' \cdot m$,

$$\gamma'_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n = \gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n.$$

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ be a clone.

Given a set X of elements of the underlying set of $\mathcal{G}$, the *subclone of $\mathcal{C}$ generated by X* is the smallest (w.r.t. inclusion of the underlying sets of the underlying graded sets) subclone $\mathcal{C}^{(X)}$ of $\mathcal{C}$ such that the underlying set of the underlying graded set of $\mathcal{C}^{(X)}$ contains X.

When X is such that $\mathcal{C}^{(X)} = \mathcal{C}$, X is a *generating set* of $\mathcal{C}$.

When X is a generating set of $\mathcal{C}$ and for any $X' \subseteq X$, $\mathcal{C}^{(X')} = \mathcal{C}^{(X)}$ implies $X' = X$, X is a *minimal generating set* of $\mathcal{C}$.

Example

It is possible to show that $\{g\}$, where $g := \{1, 2\}$ is an element of arity 2 of the underlying set of the underlying graded set of SL, is a minimal generating set of SL. For instance,

$$\{1, 3, 4\} = \gamma_{2,4} \cdot g \cdot \mathbb{1}_{1,4} \cdot \underline{\gamma_{2,4} \cdot g \cdot \mathbb{1}_{3,4} \cdot \mathbb{1}_{4,4}}.$$

A *clone congruence* on $\mathcal{C}$ is an equivalence relation $\equiv$ on the underlying set of $\mathcal{G}$ such that

- for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $[x]_\equiv \subseteq \mathcal{G} \cdot n$;
- for any $x, x' \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $y_1, y'_1, \dots, y_n, y'_n \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$, if $x \equiv x'$ and $y_i \equiv y'_i$ for all $i \in [n]$, then

$$\gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n \equiv \gamma_{n,m} \cdot x' \cdot y'_1 \cdot \cdots \cdot y'_n.$$

The *quotient* of $\mathcal{C}$ by $\equiv$ is the clone $\mathcal{C}/\equiv := \left(\mathcal{G}/\equiv, \left(\gamma'_{n,m}\right)_{n,m \in \mathbb{N}}, \left(\mathbb{1}'_{i,n}\right)_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- $\mathcal{G}/\equiv$ is the graded set satisfying, for any $n \in \mathbb{N}$, $\mathcal{G}/\equiv \cdot n = \{[x]_\equiv : x \in \mathcal{G} \cdot n\}$;
- for any $x \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, and $y_1, \dots, y_n \in \mathcal{G} \cdot m$, $m \in \mathbb{N}$,

$$\gamma'_{n,m} \cdot [x]_\equiv \cdot [y_1]_\equiv \cdot \cdots \cdot [y_n]_\equiv = [\gamma_{n,m} \cdot x \cdot y_1 \cdot \cdots \cdot y_n]_\equiv;$$

- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}'_{i,n} = [\mathbb{1}_{i,n}]_\equiv$.

10.2. Clones of pigmented words

Let X be a set. An X*-pigmented letter* is a pair (i, x) such that $i \in \mathbb{N} \setminus \{0\}$ and $x \in X$. This pair is denoted by i^x. The *value* (resp. *pigment*) of i^x is i (resp. x).

A finite sequence of X-pigmented letters is an X*-pigmented word*.

Let $\mathcal{M}$ be a **monoid** with product $\star$ and unit e.

Let the triple $\mathbf{P}{\cdot}\mathcal{M} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}{\cdot}n$ is the set of $\mathcal{M}$-pigmented words whose values of letters range all between 1 and n;
- for any $i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell} \in \mathcal{G}{\cdot}n$, $n \in \mathbb{N}$, $\ell \in \mathbb{N}$, $w_1, \dots, w_n \in \mathcal{G}{\cdot}m$, $m \in \mathbb{N}$,

$$\gamma_{n,m}{\cdot}\underbrace{i_1^{\alpha_1} \dots i_\ell^{\alpha_\ell}}{\cdot}w_1 \cdot \dots \cdot w_n := (\alpha_1 \,\bar{\star}\, w_{i_1}) \,\mathbf{.}\, \dots \,\mathbf{.}\, (\alpha_\ell \,\bar{\star}\, w_{i_\ell})$$

  where for any $\beta \in \mathcal{M}$ and any $j_1^{\beta_1} \dots j_k^{\beta_k} \in \mathcal{G}$, $k \in \mathbb{N}$,

$$\beta \,\bar{\star}\, j_1^{\beta_1} \dots j_k^{\beta_k} := j_1^{\beta \star \beta_1} \dots j_k^{\beta \star \beta_k};$$

- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n} := i^{\mathrm{e}}$.

Theorem [Clones of $\mathcal{M}$-pigmented words]

For any monoid $\mathcal{M}$, $\mathbf{P}\cdot\mathcal{M}$ is a clone.

This result comes from [S. Giraudo, Clones of pigmented words and realizations of special classes of monoids, 2026].

Examples

Let $\mathcal{M}_{\mathrm{w}}$ be the free monoid $(\{a, b\}^*, \centerdot, \epsilon)$.

The $3, 8$-projection of $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$ is $\mathbb{1}_{3,8} = 3^{\epsilon}$ and the $3, 4$-projection of $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$ is $\mathbb{1}_{3,4} = 3^{\epsilon}$. Even if they are denoted through the same $\mathcal{M}_{\mathrm{w}}$-pigmented word 3^{ϵ}, $\mathbb{1}_{3,8}$ and $\mathbb{1}_{3,4}$ are **different** elements. This remark applies also for other clones: for instance, in SL, there is a copy of the element $\{1, 3, 4\}$ for any arity $n \geqslant 4$.

Moreover, in $\mathbf{P}\cdot\mathcal{M}_{\mathrm{w}}$, we have

$$\gamma_{4,3} \cdot 2^{ab} 4^{a} 1^{\epsilon} 1^{ba} \;\cdot 2^{a} 2^{\epsilon} \;\cdot 3^{ba} 1^{a} 3^{b} \;\cdot 1^{a} \;\cdot \epsilon \;=\; 3^{ab \centerdot ba} 1^{ab \centerdot a} 3^{ab \centerdot b} \centerdot \epsilon \centerdot 2^{\epsilon \centerdot a} 2^{\epsilon \centerdot \epsilon} \centerdot 2^{ba \centerdot a} 2^{ba \centerdot \epsilon} \;=\; 3^{abba} 1^{aba} 3^{abb} 2^{a} 2^{\epsilon} 2^{baa} 2^{ba}.$$

Let $\mathcal{M}$ be a monoid.

Let $\equiv_{\mathrm{sort}}$ be the equivalence relation on the underlying set of the underlying graded set of $\mathbf{P}{\cdot}\mathcal{M}$ defined by $w \equiv_{\mathrm{sort}} w'$ if, for any $\mathcal{M}$-pigmented letter i^{α}, w and w' have both the same number of occurrences of i^{α}.

Example

The $\mathcal{M}_{\mathrm{w}}$-pigmented words $2^{ab}3^{ba}2^{b}$ and $2^{b}2^{ab}3^{ba}$ of arity 3 of $\mathbf{P}{\cdot}\mathcal{M}_{\mathrm{w}}$ are $\equiv_{\mathrm{sort}}$-equivalent.

Proposition [Sorting congruence on $\mathbf{P}{\cdot}\mathcal{M}$]

For any monoid $\mathcal{M}$, the equivalence relation $\equiv_{\mathrm{sort}}$ is a clone congruence on $\mathbf{P}{\cdot}\mathcal{M}$.

Example

In the quotient $\mathbf{P}{\cdot}\mathcal{M}_{\mathrm{w}} / \equiv_{\mathrm{sort}}$, we have

$$\gamma_{2,4} \cdot \left[1^{a}2^{b}2^{ab}\right]_{\equiv_{\mathrm{sort}}} \cdot \left[1^{aa}3^{ab}\right]_{\equiv_{\mathrm{sort}}} \cdot \left[1^{\epsilon}1^{a}1^{aa}\right]_{\equiv_{\mathrm{sort}}} = \left[1^{aaa}1^{ab}1^{aba}1^{abaa}1^{b}1^{ba}1^{baa}3^{aab}\right]_{\equiv_{\mathrm{sort}}}.$$

Let $\mathcal{M}$ be a monoid.

Let, for any $k \in \mathbb{N}$, first_k be the function sending an $\mathcal{M}$-pigmented word w to the subword of w obtained by deleting all occurrences of $\mathcal{M}$-pigmented letters having $i \in \mathbb{N} \setminus \{0\}$ as value when there are k or more occurrences of $\mathcal{M}$-pigmented letters of value i on the left.

Let also $\equiv_{\mathrm{first}_k}$ be the kernel of first_k.

Example

For any $\alpha_1, \dots, \alpha_9 \in \mathcal{M}$,

$$\mathrm{first}_2 \cdot 1^{\alpha_1} 2^{\alpha_2} 1^{\alpha_3} 3^{\alpha_4} 1^{\alpha_5} 3^{\alpha_6} 2^{\alpha_7} 4^{\alpha_8} 3^{\alpha_9} = 1^{\alpha_1} 2^{\alpha_2} 1^{\alpha_3} 3^{\alpha_4} 3^{\alpha_6} 2^{\alpha_7} 4^{\alpha_8}.$$

Proposition [First congruence on $\mathbf{P} \cdot \mathcal{M}$]

For any monoid $\mathcal{M}$ and $k \in \mathbb{N}$, the equivalence relation $\equiv_{\mathrm{first}_k}$ is a clone congruence on $\mathbf{P} \cdot \mathcal{M}$.

Example

In the quotient $\mathbf{P} \cdot \mathcal{M}_{\mathrm{w}} / \equiv_{\mathrm{first}_1}$, we have for instance

$$\gamma_{3,3} \cdot \left[2^{ba} 1^{bab} 3^{b}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[2^{b} 1^{a}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[2^{aa} 3^{a}\right]_{\equiv_{\mathrm{first}_1}} \cdot \left[3^{aa} 1^{a} 2^{\epsilon}\right]_{\equiv_{\mathrm{first}_1}} = \left[2^{baaa} 3^{baa} 1^{baba}\right]_{\equiv_{\mathrm{first}_1}}.$$

When $\mathcal{M}$ is the **trivial monoid** $\mathbf{E} := (\{e\}, \star, e)$, the elements of the underlying set of the underlying graded set of $\mathbf{P}\cdot\mathbf{E}$ are *monochrome pigmented words*. In this case, we denote simply by $i_1 \dots i_\ell$, $\ell \in \mathbb{N}$, the monochrome pigmented word $i_1^{e} \dots i_\ell^{e}$.

Examples

In $\mathbf{P}\cdot\mathbf{E}$, we have

$$\gamma_{4,3} \cdot 2411 \cdot 22 \cdot 313 \cdot 1 \cdot \epsilon = 313.\epsilon.22.22 = 3132222.$$

In $\mathbf{P}\cdot\mathbf{E}/\equiv_{\mathrm{sort}}$, we have

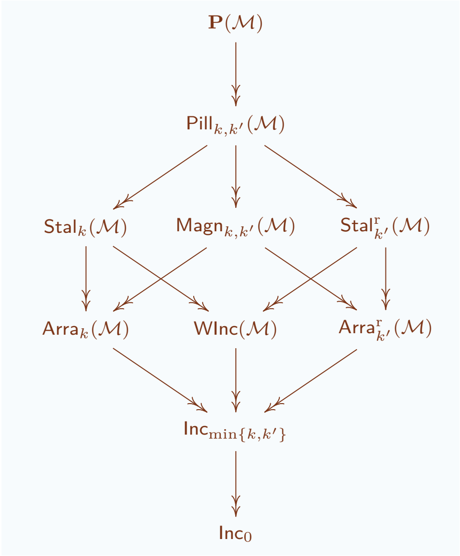
$$\gamma_{2,4} \cdot [122]_{\equiv_{\mathrm{sort}}} \cdot [13]_{\equiv_{\mathrm{sort}}} \cdot [111]_{\equiv_{\mathrm{sort}}} = [11111113]_{\equiv_{\mathrm{sort}}}.$$

In $\mathbf{P}\cdot\mathbf{E}/\equiv_{\mathrm{first}_1}$, we have

$$\gamma_{3,3} \cdot [213]_{\equiv_{\mathrm{first}_1}} \cdot [21]_{\equiv_{\mathrm{first}_1}} \cdot [23]_{\equiv_{\mathrm{first}_1}} \cdot [312]_{\equiv_{\mathrm{first}_1}} = [231]_{\equiv_{\mathrm{first}_1}}.$$

It is possible to build a **hierarchy of clones** by considering quotients $\mathbf{P}\cdot\mathcal{M}$, where $\mathcal{M}$ is a monoid, by using the clone congruences $\equiv_{\mathrm{sort}}$ and $\equiv_{\mathrm{first}_k}$, $k \in \mathbb{N}$, their intersections, and their reversions.

Here is the diagram, where arrows are surjective clone morphisms (without further details):



10.3. Free clones and presentations

For any signature $\mathcal{S}$, let the triple $\mathcal{FC}\cdot\mathcal{S} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m\in\mathbb{N}}, (\mathbb{1}_{i,n})_{n\in\mathbb{N}, i\in[n]}\right)$ such that

- $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}\cdot n$ is the set of labeled $\mathcal{S}$-terms $\mathfrak{t}$ such that $\mathrm{rk_v}\cdot\mathfrak{t} \leqslant n$;
- for any $\mathfrak{t} \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathcal{G}\cdot m$, $m \in \mathbb{N}$,

$$\gamma_{n,m}\cdot\mathfrak{t}\cdot\mathfrak{t}_1\cdot \cdots \cdot\mathfrak{t}_n := \mathfrak{t}[\mathfrak{t}_1, \dots, \mathfrak{t}_n];$$

- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is the labeled $\mathcal{S}$-term consisting in one leaf decorated by i.

Let $\iota : \mathcal{S} \to \mathfrak{T}\cdot\mathcal{S}\cdot\underbrace{\mathbb{N} \setminus \{0\}}$ be the function such that for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, $\iota\cdot c$ is the labeled $\mathcal{S}$-term $c1\dots n$.

Theorem [Free clones]

For any signature $\mathcal{S}$, $\mathcal{FC}\cdot\mathcal{S}$ is a clone. Moreover, for any clone $\mathcal{C}$ and any rank-preserving function f from the underlying set of $\mathcal{S}$ to the underlying set of the underlying graded set of $\mathcal{C}$, there exists a unique clone morphism ϕ from $\mathcal{FC}\cdot\mathcal{S}$ to $\mathcal{C}$ such that $f = \phi \circ \iota$.

The class of clones together with clone morphisms forms a category. Theorem [Free clones] says that for any signature $\mathcal{S}$, $\mathcal{FC}\cdot\mathcal{S}$ is a free object in this category.

Let $\mathcal{C}$ be a clone.

A *presentation* of $\mathcal{C}$ is an **equational presentation** $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ such that $\mathcal{C}$ is isomorphic to $\mathcal{FC} \cdot \mathcal{S} / \equiv$ where $\equiv$ is the smallest clone congruence on $\mathcal{FC} \cdot \mathcal{S}$ containing $\sim$.

Example

The clone SL admits the equational presentation $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ as presentation, where $\mathcal{S}$ is the signature containing a binary constant $\wedge$ and $\sim$ is defined by $\wedge \underline{\wedge 12} 3 \sim \wedge 1 \underline{\wedge 23}$, $\wedge 12 \sim \wedge 21$, and $\wedge 11 \sim 1$.

A clone $\mathcal{C}$ can have different presentations, even on **non-isomorphic signatures**.

Example

Let the equational presentation $(\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ is the signature containing a binary constant $\wedge$ and a ternary constant t, and $\sim'$ is defined by $\wedge \underline{\wedge 12} 3 \sim' \wedge 1 \underline{\wedge 23}$, $\wedge 12 \sim' \wedge 21$, $\wedge 11 \sim' 1$, and $t123 \sim' \wedge \underline{\wedge 12} 3$.

Then, $\mathcal{E}'$ is another presentation of SL.

Let $\mathcal{C} := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ be a clone.

For any $n \in \mathbb{N}$, the *$\mathcal{C}, n$-evaluation function* is the function defined, for any $\mathcal{G}, [n]$-term $\mathfrak{t}$, by

$$\mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t} := \begin{cases} \mathbb{1}_{i,n} & \text{if } \mathfrak{t} = i \text{ with } i \in [n], \\ \gamma_{m,n} \cdot c \cdot \llcorner \mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t}_1 \lrcorner \cdot \ \cdots \ \cdot \llcorner \mathrm{ev}_{\mathcal{C},n} \cdot \mathfrak{t}_m \lrcorner & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \dots \mathfrak{t}_m, c \in \mathcal{G} \cdot m, m \in \mathbb{N}, \mathfrak{t}_i \in \mathfrak{T} \cdot \mathcal{G} \cdot [n], i \in [m]. \end{cases}$$

Example

In SL, we have

$$\mathrm{ev}_{\mathrm{SL},4} \cdot \{1,3\} \llcorner \{2,4\} 1324 \lrcorner 14 = \{3,4\}.$$

Proposition [Raw presentations of clones]

Let $\mathcal{C}$ be a clone having $\mathcal{G}$ as underlying graded set. The equational presentation $(\mathcal{G}, \mathbb{N} \setminus \{0\}, \sim)$ is a presentation of $\mathcal{C}$, where $\sim$ is defined by $\mathfrak{t} \sim \mathfrak{t}'$ if $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{G} \cdot [n]$, $n \in \mathbb{N}$, and $(\mathfrak{t}, \mathfrak{t}') \in \mathrm{Ker} \cdot \mathrm{ev}_{\mathcal{C},n}$.

The presentation of $\mathcal{C}$ described by Proposition [Raw presentations of clones] is the *raw presentation* of $\mathcal{C}$.

10.4. Algebras over clones

For any set A, let the triple $\mathrm{End}\cdot A := \left(\mathcal{G}, (\gamma_{n,m})_{n,m \in \mathbb{N}}, (\mathbb{1}_{i,n})_{n \in \mathbb{N}, i \in [n]}\right)$ such that

- $\mathcal{G}$ is the graded set such that for any $n \in \mathbb{N}$, $\mathcal{G}\cdot n$ is the set of n-operations on A;
- for any $\phi \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, $\psi_1, \dots, \psi_n \in \mathcal{G}\cdot m$, $m \in \mathbb{N}$, and $a_1, \dots, a_m \in A$,
$$\gamma_{n,m}\cdot\phi\cdot\psi_1\cdot \cdots \cdot\psi_n\cdot a_1\cdot \cdots \cdot a_m := \phi\cdot\lfloor\psi_1\cdot a_1\cdot \cdots \cdot a_m\rfloor\cdot \cdots \cdot\lfloor\psi_n\cdot a_1\cdot \cdots \cdot a_m\rfloor;$$
- for any $n \in \mathbb{N}$ and $i \in [n]$, $\mathbb{1}_{i,n}$ is the n-operation on A defined, for any $a_1, \dots, a_n \in A$, by
$$\mathbb{1}_{i,n}\cdot a_1\cdot \cdots \cdot a_n := a_i.$$

Proposition [Clones of endomorphisms]

For any set A, $\mathrm{End}\cdot A$ is a clone.

The clone $\mathrm{End}\cdot A$ is the *clone of endomorphisms on* A.

Exercise ●●●○○

Prove Proposition [Clones of endomorphisms].

Let $\mathcal{C}$ be a clone and A be a set.

An *algebra over $\mathcal{C}$ on A* is a $\mathcal{G}$-algebra $(A, \mathcal{G}, \mathrm{op})$ such that

- $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$;
- there exists a clone morphism ϕ from $\mathcal{C}$ to $\mathrm{End}\cdot A$ such that for any $x \in \mathcal{G}\cdot n$, $n \in \mathbb{N}$, $\mathrm{op}\cdot x$ is the n-operation $\phi\cdot x$ on A.

Observe that if X is a generating set of $\mathcal{C}$, then ϕ is uniquely determined by the images by ϕ on X. Hence, to define an algebra over $\mathcal{C}$, it is enough to describe $\mathrm{op}\cdot x$ for any $x \in X$.

Example

Consider the clone SL and the set $A := \mathbb{Z}$.
Let ϕ be the clone morphism from SL to $\mathrm{End}\cdot A$ defined, for any $S \in \mathcal{P}\cdot[n] \setminus \emptyset$, $n \in \mathbb{N}$, by

$$\phi\cdot S\cdot a_1\cdot \cdots \cdot a_n := \max\{a_i : i \in S\}.$$

This defines an algebra $(A, \mathcal{G}, \mathrm{op})$ over SL on A where $\mathcal{G}$ is the underlying graded set of SL. We have for instance

$$\mathrm{op}\cdot\{1,3\}\cdot -4\cdot 3\cdot 2 = \max\{-4, 2\} = 2.$$

Since $\{\{1,2\}\}$ is a generating set of SL, the operations $\mathrm{op}\cdot S$ are uniquely determined for any $S \in \mathcal{P}\cdot[n] \setminus \emptyset$, $n \in \mathbb{N}$, by $\mathrm{op}\cdot\{1,2\}\cdot a_1\cdot a_2 = \max\{a_1, a_2\}$, where $a_1, a_2 \in A$.

Theorem [Algebras over clones and varieties]

Let $\mathcal{C}$ be a clone having $\mathcal{G}$ as underlying graded set.

1. The class of algebras over $\mathcal{C}$ is a variety of $\mathcal{G}$-algebras.
2. This variety of $\mathcal{C}$-algebras is the variety of the raw presentation of $\mathcal{C}$.

Theorem [Algebras over clones and varieties] relies on Theorem [Birkhoff's Variety Theorem] and says that clones, through their algebras, describe varieties of algebras.

Example

Consider the algebras over the clone SL.

Since the set $\{g\}$, where $g := \{1, 2\}$, is a minimal generating set of SL, to define an algebra $\mathcal{A} := (A, \mathcal{G}, \mathrm{op})$ over SL where A is a set and $\mathcal{G}$ is the underlying graded set of SL, it is enough to define $\mathrm{op}{\cdot}g$.

We can check that, due to the definition of SL, $\mathrm{op}{\cdot}g$ is an associative, commutative, and idempotent 2-operation on A. Therefore, $\mathcal{A}$ is a semilattice.

Definition

Let $\mathcal{E}$ and $\mathcal{E}'$ be two equational presentations having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables. If there exists a clone $\mathcal{C}$ which admits both $\mathcal{E}$ and $\mathcal{E}'$ as presentations, then $\mathcal{E}$ and $\mathcal{E}'$ are *equivalent*.

Definition

Let $\mathcal{V}$ and $\mathcal{V}'$ be two varieties. If there exist two equivalent equational presentations $\mathcal{E}$ and $\mathcal{E}'$ such that the class of algebras over $\mathcal{E}$ (resp. $\mathcal{E}'$) is $\mathcal{V}$ (resp. $\mathcal{V}'$), then $\mathcal{V}$ and $\mathcal{V}'$ are *equivalent*.

Thus, clones encode equivalence classes of varieties through their algebras.

A natural question about a variety $\mathcal{V}$ consists in exhibiting a variety $\mathcal{V}'$ such that $\mathcal{V}$ and $\mathcal{V}'$ are equivalent and $\mathcal{V}'$ is the simplest possible one. By ‘‘simplest’’, we mean a variety which is described by an equational presentation $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ such that

- ☐ $\mathcal{S}$ is the smallest possible, for a certain size notion on signatures;
- ☐ and/or $\sim$ is the smallest possible, for a certain size notion on elementary identity relations.

Exercise ●●●○○

Prove that the equational presentations **Groups** and **PHeaps** are equivalent.

Exercise ●●●●○

Let $\mathbf{AG_1} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be the usual equational presentation of *abelian groups*, where $\mathcal{S}$ contains a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \lfloor m12 \rfloor 3 \sim m1 \lfloor m23 \rfloor$, $me1 \sim 1$, $m \lfloor i1 \rfloor 1 \sim e$, and $m12 \sim m21$.

Let also the equational presentation $\mathbf{AG_2} := (\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ contains one nullary constant c and one binary constant d, and $\sim'$ is defined by $d11 \sim' c$ and $d1 \lfloor d2 \lfloor d3 \lfloor d12 \rfloor \rfloor \rfloor \sim' 3$.

Prove that $\mathbf{AG_1}$ and $\mathbf{AG_2}$ are equivalent.

Let $\mathcal{C}$ and $\mathcal{C}'$ be two clones, and ϕ be a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$.

Let $\mathcal{A}' := (A, \mathcal{G}', \mathrm{op}')$ be an algebra over $\mathcal{C}'$ on a set A. Let $\phi \cdot \mathcal{A}' := (A, \mathcal{G}, \mathrm{op})$ be the triple such that

- $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$;
- for any $g \in \mathcal{G} \cdot n$, $n \in \mathbb{N}$, $\mathrm{op} \cdot g$ is the n-operation on A defined, for any $a_1, \dots, a_n \in A$, by

$$\mathrm{op} \cdot g \cdot a_1 \cdot \dots \cdot a_n := \mathrm{op}' \cdot \llcorner \phi \cdot g \lrcorner \cdot a_1 \cdot \dots \cdot a_n.$$

Theorem [Clone morphisms and algebras]

Let $\mathcal{C}$ and $\mathcal{C}'$ be two clones, and ϕ be a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$. If $\mathcal{A}'$ is an algebra over $\mathcal{C}'$, then $\phi \cdot \mathcal{A}'$ is an algebra over $\mathcal{C}$.

Theorem [Clone morphisms and algebras] says that a clone morphism from a clone $\mathcal{C}$ to a clone $\mathcal{C}'$ gives rise to a transformation of an algebra over $\mathcal{C}'$ to an algebra over $\mathcal{C}$. This transformation can be described as a functor from the category of algebras over $\mathcal{C}'$ to the category of algebras over $\mathcal{C}$.

Example

Let the equational presentation $\mathcal{E} := (\mathrm{MagC}, \mathbb{N} \setminus \{0\}, \sim)$ where $\sim$ satisfies $m \llcorner \underline{m12} \lrcorner 3 \sim m1 \llcorner \underline{m23} \lrcorner$, $m \llcorner \underline{m12} \lrcorner 1 \sim m12$, $m1c \sim 1$, and $mc1 \sim 1$.

This is the equational presentation **LBMonoids** of *left-regular band monoids*.

Let $\mathcal{C} := \mathcal{FC}{\cdot}\mathrm{MagC}/\equiv$ where $\equiv$ is the smallest clone congruence containing $\sim$.

Let the equational presentation $\mathcal{E}' := (\mathrm{MagC}, \mathbb{N} \setminus \{0\}, \sim')$ where $\sim'$ satisfies $m \llcorner \underline{m12} \lrcorner 3 \sim' m1 \llcorner \underline{m23} \lrcorner$, $m1c \sim' 1$, $mc1 \sim' 1$, $m12 \sim' m21$, and $m11 \sim' 1$.

This is the equational presentation **BSLattices** of **bounded semilattices**.

Let $\mathcal{C}' := \mathcal{FC}{\cdot}\mathrm{MagC}/\equiv'$ where $\equiv'$ is the smallest clone congruence containing $\sim'$.

Let ϕ be the function from the underlying set of the underlying graded set of $\mathcal{C}$ to the underlying set of the underlying graded set of $\mathcal{C}'$ sending $[m]_{\equiv}$ to $[m]_{\equiv'}$ and $[c]_{\equiv}$ to $[c]_{\equiv'}$. It is possible to show that ϕ can be uniquely extended as a clone morphism from $\mathcal{C}$ to $\mathcal{C}'$.

By Theorem [Clone morphisms and algebras], if $\mathcal{A}$ is a bounded semilattice, then $\mathcal{A}$ is also a left-regular band monoid.

10.5. Clone realizations of varieties

Let $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be an equational presentation.

A clone $\mathcal{C}$ admitting $\mathcal{E}$ as presentation is a *clone realization* of $\mathcal{E}$.

Let $\equiv$ be the smallest clone congruence on $\mathcal{FC}{\cdot}\mathcal{S}$ containing $\sim$. Fix a clone isomorphism $\bar{\phi}$ from $\mathcal{FC}{\cdot}\mathcal{S}/\equiv$ to $\mathcal{C}$, and set $\phi := \bar{\phi} \circ \pi \circ \iota$ where π is the canonical projection function from $\mathcal{FC}{\cdot}\mathcal{S}$ to $\mathcal{FC}{\cdot}\mathcal{S}/\equiv$.

The function ϕ is an *interpretation function* of $\mathcal{E}$ in $\mathcal{C}$.

For any $n \in \mathbb{N}$, let $\mathrm{ev}^{\phi}_{\mathcal{C},n}$ be the function defined, for any $\mathcal{S}, [n]$-term $\mathfrak{t}$, by $\mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t} := \mathrm{ev}_{\mathcal{C},n}{\cdot}\mathfrak{s}$, where $\mathfrak{s}$ is the $\mathcal{G}, [n]$-term obtained from $\mathfrak{t}$ by replacing each constant decoration c by $\phi{\cdot}c$, and $\mathcal{G}$ is the underlying graded set of $\mathcal{C}$.

Theorem [Clone realizations and word problems]

Let $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ be an equational presentation, $\mathcal{C}$ be a clone realization of $\mathcal{E}$, and ϕ be an interpretation function of $\mathcal{E}$ in $\mathcal{C}$. Then, for any $n \in \mathbb{N}$ and any $\mathcal{S}, [n]$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \equiv_{\mathcal{E}} \mathfrak{t}'$ iff $\mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t} = \mathrm{ev}^{\phi}_{\mathcal{C},n}{\cdot}\mathfrak{t}'$.

Example

Let the clone $\mathsf{Arra}_1 := \mathbf{P}\cdot\mathbf{E}/ \equiv_{\mathrm{first}_1}$.
Let us identify each $\equiv_{\mathrm{first}_1}$-class $[w]_{\equiv_{\mathrm{first}_1}}$ of monochrome pigmented words by the unique monochrome pigmented word of minimal length in $[w]_{\equiv_{\mathrm{first}_1}}$. For instance, we identify $[2312]_{\equiv_{\mathrm{first}_1}}$ with 231 since $[2312]_{\equiv_{\mathrm{first}_1}} = \{231, 2231, 2331, 2311, 2311, 2312, \dots\}$. Under this identification, the underlying graded set $\mathcal{G}$ of Arra_1 is such that $\mathcal{G}\cdot n$, $n \in \mathbb{N}$, is the set of word on $[n]$ having at most one occurrence of each $i \in [n]$.
We have for instance

$$\gamma_{5,6}\cdot 3152\cdot 213\cdot 6132\cdot 3215\cdot 2\cdot 6512 = 32156.$$

It is possible to show that Arra_1 is a clone realization of the equational presentation $\mathbf{LBMonoids}$ of left-regular band monoids. Let us assume this property.
Now, let ϕ be the function such that $\phi\cdot m = 12$ and $\phi\cdot c = \epsilon$. This is an interpretation function of $\mathbf{LBMonoids}$ in Arra_1.
Let the $\mathrm{MagC}, \mathbb{N} \setminus \{0\}$-terms $\mathfrak{t} := m\lfloor m2\lfloor m21\rfloor\rfloor\lfloor m\lfloor m2c\rfloor 3\rfloor$ and $\mathfrak{t}' := m\lfloor m22\rfloor\lfloor m13\rfloor$. Since

$$\mathrm{ev}^{\phi}_{\mathsf{Arra}_1,3}\cdot\mathfrak{t} = \mathrm{ev}_{\mathsf{Arra}_1,3}\cdot\lfloor 12\lfloor 12\,2\lfloor 12\,2\,1\rfloor\rfloor\lfloor 12\lfloor 12\,2\,\epsilon\rfloor 3\rfloor\rfloor = 213$$

and

$$\mathrm{ev}^{\phi}_{\mathsf{Arra}_1,3}\cdot\mathfrak{t}' = \mathrm{ev}_{\mathsf{Arra}_1,3}\cdot\lfloor 12\lfloor 1222\rfloor\lfloor 1213\rfloor\rfloor = 213,$$

this shows that $\mathfrak{t} \equiv_{\mathbf{LBMonoids}} \mathfrak{t}'$.

Exercise ○○○○○○

Let $\mathcal{M}$ be a monoid with product $\star$ and unit e.

Let $\mathcal{E}_{\mathcal{M}} := (\mathcal{S}_{\mathcal{M}}, \mathbb{N} \setminus \{0\}, \sim_{\mathcal{M}})$ be the equational presentation of $\mathcal{M}$-*pigmented monoids* where $\mathcal{S}_{\mathcal{M}}$ is the signature containing a nullary constant u, unary constants p_α, $\alpha \in \mathcal{M}$, and a binary constant $\star$, and $\sim_{\mathcal{M}}$ is defined by

$$\star \underline{\star 1 2} 3 \sim_{\mathcal{M}} \star 1 \underline{\star 2 3},$$

$$\star \mathrm{u} 1 \sim_{\mathcal{M}} 1, \qquad \star 1 \mathrm{u} \sim_{\mathcal{M}} 1,$$

$$\mathrm{p}_\alpha \underline{\star 1 2} \sim_{\mathcal{M}} \star \underline{\mathrm{p}_\alpha 1}\, \underline{\mathrm{p}_\alpha 2},$$

$$\mathrm{p}_\alpha \mathrm{u} \sim_{\mathcal{M}} \mathrm{u},$$

$$\mathrm{p}_{\alpha_1} \underline{\mathrm{p}_{\alpha_2} 1} \sim_{\mathcal{M}} \mathrm{p}_{\alpha_1 \star \alpha_2} 1,$$

$$\mathrm{p}_{\mathrm{e}} 1 \sim_{\mathcal{M}} 1,$$

for any $\alpha, \alpha_1, \alpha_2 \in \mathcal{M}$.

Show that $\mathbf{P} \cdot \mathcal{M}$ is a clone realization of $\mathcal{E}_{\mathcal{M}}$.

10.6. Tietze rules

The *Tietze rules* consist in the following four rules allowing us to transform an equational presentation $\mathcal{E} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ into another one:

1. **[Constant Adding]** Transform $\mathcal{E}$ into $(\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ such that there exists a new constant c and $n \in \mathbb{N}$ such that $\mathcal{S}' \cdot n = \mathcal{S} \cdot n \sqcup \{c\}$, for any $m \in \mathbb{N} \setminus \{n\}$, $\mathcal{S}' \cdot m = \mathcal{S} \cdot m$, and $\sim' = \sim \cup \{(c\, 1 \dots n, \mathfrak{t})\}$ where $\mathfrak{t}$ is an $\mathcal{S}, [n]$-term;
2. **[Constant Deleting]** Transform $\mathcal{E}$ into $\mathcal{E}'$ if $\mathcal{E}$ is obtained from $\mathcal{E}'$ through an application of the Constant Adding rule;
3. **[Elementary Identity Adding]** Transform $\mathcal{E}$ into $(\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim')$ such that $\sim' = \sim \cup \{(\mathfrak{t}_1, \mathfrak{t}_2)\}$ where $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are two $\mathcal{S}, \mathbb{N} \setminus \{0\}$-terms satisfying $(\mathfrak{t}_1, \mathfrak{t}_2) \notin \sim$ and $\mathfrak{t}_1 \approx_{\mathcal{E}} \mathfrak{t}_2$;
4. **[Elementary Identity Deleting]** Transform $\mathcal{E}$ into $\mathcal{E}'$ if $\mathcal{E}$ is obtained from $\mathcal{E}'$ through an application of the Elementary Identity Adding rule.

Two equational presentations $\mathcal{E}$ and $\mathcal{E}'$ having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables are *Tietze equivalent* if it is possible to transform $\mathcal{E}$ into $\mathcal{E}'$ by means of the iterative application of the Tietze rules.

Example 1/6

Recall that **Groups** is the equational presentation $\mathbf{Groups} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \sim)$ where $\mathcal{S}$ is the signature containing a nullary constant e, a unary constant i, and a binary constant m, and $\sim$ is defined by $m \llcorner m12 \lrcorner 3 \sim m1 \llcorner m23 \lrcorner$, $me1 \sim 1$, $m1e \sim 1$, $m \llcorner i1 \lrcorner 1 \sim e$, and $m1 \llcorner i1 \lrcorner \sim e$.

Recall moreover that **PHeaps** is the equational presentation $\mathbf{PHeaps} := (\mathcal{S}', \mathbb{N} \setminus \{0\}, \sim')$ where $\mathcal{S}'$ is the signature containing a nullary constant e' and a ternary constant p', and $\sim'$ is defined by $p'112 \sim' 2$, $p'122 \sim' 1$, and $p' \llcorner p'123 \lrcorner 45 \sim' p'12 \llcorner p'345 \lrcorner$.

Let us show that **PHeaps** and **Groups** are Tietze equivalent.

Set $P_0 := \mathbf{PHeaps}$.

Starting from P_0, we apply three times the Constant Adding rule:

1. we add a new nullary constant e together with the elementary identity
$$e \sim_0 e';$$
2. we add a new unary constant i together with the elementary identity
$$i1 \sim_0 p'e1e;$$
3. we add a new binary constant m together with the elementary identity
$$m12 \sim_0 p'1e2.$$

Let P_1 be the obtained equational presentation and let $\equiv_1 := \approx_{P_1}$.

Example 2/6

In P_1, the five identities of **Groups** are derivable:

$$m \llcorner m12 \lrcorner 3 \equiv_1 p' \llcorner p'1e2 \lrcorner e3 \equiv_1 p'1e \llcorner p'2e3 \lrcorner \equiv_1 m1 \llcorner m23 \lrcorner,$$

$$me1 \equiv_1 p'ee1 \equiv_1 1,$$

$$m1e \equiv_1 p'1ee \equiv_1 1,$$

$$m \llcorner i1 \lrcorner 1 \equiv_1 p' \llcorner p'e1e \lrcorner e1 \equiv_1 p'e1 \llcorner p'ee1 \lrcorner \equiv_1 p'e11 \equiv_1 e,$$

$$m1 \llcorner i1 \lrcorner \equiv_1 p'1e \llcorner p'e1e \lrcorner \equiv_1 \llcorner p'1ee \lrcorner 1e \equiv_1 p'11e \equiv_1 e.$$

Hence, by five applications of the Elementary Identity Adding rule, we add the five elementary identities of **Groups**.

Let P_2 be the obtained equational presentation and let $\equiv_2 := \approx_{P_2}$.

Example 3/6

In P_2, we have

$$e' \equiv_2 e,$$

and

$$m\underline{m1\underline{i2}}3 \equiv_2 p'\underline{p'1e\underline{p'e2e}}e3 \equiv_2 p'1e\underline{p'\underline{p'e2e}e3} \equiv_2 p'1e\underline{p'e2\underline{p'ee3}}$$
$$\equiv_2 p'1e\underline{p'e23} \equiv_2 \underline{p'1ee}23 \equiv_2 p'123.$$

Hence, by two further applications of the Elementary Identity Adding rule, we add the two elementary identities

$$e' \sim_2 e$$

and

$$p'123 \sim_2 m\underline{m1\underline{i2}}3.$$

Let P_3 be the obtained equational presentation and let $\equiv_3 := \approx_{P_3}$.

Example 4/6

Starting from P_3, we now delete the three original identities of **PHeaps**.

This is possible because in P_3, by removing respectively these identities, we still have

$$p'112 \equiv_3 m\underbracket{m1\underbracket{i1}}2 \equiv_3 me2 \equiv_3 2,$$

$$p'122 \equiv_3 m\underbracket{m1\underbracket{i2}}2 \equiv_3 m1\underbracket{m\underbracket{i2}2} \equiv_3 m1e \equiv_3 1,$$

and

$$p'\underbracket{p'123}45 \equiv_3 m\underbracket{m\underbracket{m\underbracket{m1\underbracket{i2}}3}\underbracket{i4}}5 \equiv_3 m\underbracket{m\underbracket{m1\underbracket{i2}}3}\underbracket{m\underbracket{i4}5}$$

$$\equiv_3 m\underbracket{m1\underbracket{i2}}\underbracket{m3\underbracket{m\underbracket{i4}5}} \equiv_3 m\underbracket{m1\underbracket{i2}}\underbracket{m\underbracket{m3\underbracket{i4}}5} \equiv_3 p'12\underbracket{p'345}.$$

Therefore, by three applications of the Elementary Identity Deleting rule, we delete

$$p'112 \sim_3 2,$$

$$p'122 \sim_3 1,$$

and

$$p'\underbracket{p'123}45 \sim_3 p'12\underbracket{p'345}.$$

Let P_4 be the obtained equational presentation and let $\equiv_4 := \approx_{P_4}$.

Example 5/6

Starting from P_4, we now delete the three identities

$$e \sim_4 e',$$

$$i1 \sim_4 p'e1e,$$

and

$$m12 \sim_4 p'1e2.$$

This is possible because in P_4, by removing respectively these identities, we still have

$$e \quad \equiv_4 \quad e',$$

$$p'e1e \quad \equiv_4 \quad m\llcorner me \llcorner i1 \lrcorner \lrcorner e \quad \equiv_4 \quad m \llcorner i1 \lrcorner e \quad \equiv_4 \quad i1,$$

and, since

$$ie \quad \equiv_4 \quad m \llcorner ie \lrcorner e \quad \equiv_4 \quad e,$$

we have

$$p'1e2 \quad \equiv_4 \quad m \llcorner m1 \llcorner ie \lrcorner \lrcorner 2 \quad \equiv_4 \quad m \llcorner m1e \lrcorner 2 \quad \equiv_4 \quad m12.$$

Therefore, by three applications of the Elementary Identity Deleting rule, we delete these three identities.

Let P_5 be the obtained equational presentation.

Example 6/6

In P_5, the constant p' occurs only in

$$p'123 \sim_5 m \llcorner m1 \llcorner i2 \lrcorner \lrcorner 3.$$

Hence, by the Constant Deleting rule, we delete p'.

In the obtained presentation, the constant e' occurs only in

$$e' \sim_5 e.$$

Hence, by the Constant Deleting rule, we delete e'.

The resulting equational presentation is exactly **Groups**.

Therefore, **PHeaps** and **Groups** are Tietze equivalent.

Proposition [Soundness of the Tietze rules]

If an equational presentation $\mathcal{E}'$ is obtained from an equational presentation $\mathcal{E}$ by one Tietze rule, then $\mathcal{E}$ and $\mathcal{E}'$ are equivalent.

An equational presentation $\mathcal{E} := (\mathcal{S}, \mathsf{V}, \sim)$ is *finite* if the underlying set of $\mathcal{S}$ is finite and $\sim$ is finite.

Theorem [Tietze rules and equivalence of equational presentations]

Two **finite** equational presentations $\mathcal{E}$ and $\mathcal{E}'$ having both $\mathbb{N} \setminus \{0\}$ as underlying set of variables are equivalent iff $\mathcal{E}$ and $\mathcal{E}'$ are Tietze equivalent.

Examples

The equational presentations **Groups** and **PHeaps** are finite. Moreover, by the previous example, **Groups** and **PHeaps** are Tietze equivalent. Hence, by Theorem [Tietze rules and equivalence of equational presentations], **Groups** and **PHeaps** are equivalent.

Therefore, the variety of groups and the variety of pointed heaps are equivalent.

This shows that two varieties can be equivalent even when it appears that they are presented on different signatures.

11. Term rewriting programming

11.1. Computing with TRSs and rewrite strategies

TRSs can be used as computing frameworks. Indeed, if $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ is a TRS,

- rewrite relations $\Rightarrow$ describe how to make a **computation progress**: if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, where $\mathfrak{t}$ and $\mathfrak{t}'$ are $\mathcal{S}, \mathsf{V}$-terms, $\mathfrak{t}'$ is the *next step of computation* from $\mathfrak{t}$;
- normal forms of $\mathcal{T}$ are the **values** of the computation: if $\mathfrak{t}' \in \mathfrak{t}^{\Rightarrow}$ where $\mathfrak{t}$ is an $\mathcal{S}, \mathsf{V}$-term, then $\mathfrak{t}'$ is a *computation result* of $\mathfrak{t}$.

Thus, TRSs form a **programming paradigm** where $\mathcal{S}$ encodes what are the **syntactically valid expressions** and V encodes what are the **allowed variables**.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing three nullary constants $\mathbf{true}$, $\mathbf{false}$, and $\mathbf{zero}$, two unary constants $\mathbf{succ}$ and even, and a ternary constant if.

Let $\to$ be the elementary rewrite relation defined by $\mathsf{if}\, \mathbf{true}\, 1\, 2 \to 1$, $\mathsf{if}\, \mathbf{false}\, 1\, 2 \to 2$, $\mathsf{even}\, \mathbf{zero} \to \mathbf{true}$, $\mathsf{even}\, \underline{\mathbf{succ}\, \mathbf{zero}} \to \mathbf{false}$, and $\mathsf{even}\, \underline{\mathbf{succ}\, \underline{\mathbf{succ}\, 1}} \to \mathsf{even}\, 1$.

Then, the $\mathcal{S}, \mathbb{N}$-term $\mathfrak{t} := \mathsf{if}\, \underline{\mathsf{even}\, \underline{\mathbf{succ}\, \underline{\mathbf{succ}\, \underline{\mathbf{succ}\, \mathbf{zero}}}}}\, \mathfrak{t}_1\, \mathfrak{t}_2$ encodes the expression

`if even 3 then t1 else t2.`

Its computation in $\mathcal{T}$ is

$$\mathfrak{t} \;\Rightarrow\; \mathsf{if}\, \underline{\mathsf{even}\, \underline{\mathbf{succ}\, \mathbf{zero}}}\, \mathfrak{t}_1\, \mathfrak{t}_2 \;\Rightarrow\; \mathsf{if}\, \mathbf{false}\, \mathfrak{t}_1\, \mathfrak{t}_2 \;\Rightarrow\; \mathfrak{t}_2.$$

Hence, a normal form of $\mathfrak{t}_2$ is a computation result of $\mathfrak{t}$.

There are two main approaches:

1. a **program** is encoded by a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ where the elementary rewrite relation $\rightarrow$ represents a **set of instructions**, and **input data** is an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$;
2. a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ is fixed once and for all, and an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ encodes **both a program and input data**.

Here are some remarks about these two approaches when working with a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$:

- when $\mathcal{T}$ is **not terminating**, for some input, the computation can **diverge**;
- when $\mathcal{T}$ is **not confluent**, for some input, the computation can be **non-deterministic**;
- when $\mathfrak{t}$ is a terminating $\mathcal{S}, \mathsf{V}$-term, the computation of a result from $\mathfrak{t}$ in $\mathcal{T}$ may require **more or fewer computation steps**.

Definition

A *rewrite strategy* on an ARS $\mathcal{A} := (X, \Rightarrow)$ is a partial function s from X to X such that

1. for any $x \in X$, $s \cdot x$ is defined iff x is not a normal form of $\mathcal{A}$;
2. if $s \cdot x$ is defined, then $x \Rightarrow s \cdot x$.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

For any strategy s on $\mathcal{A}$ and any $x, x' \in X$, we write $x \Rightarrow_s x'$ for the fact that $x' = s \cdot x$. In particular, $\Rightarrow_s$ is a rewrite relation on X. Moreover, for any $x \in X$, let $s^* \cdot x := x^{\Rightarrow_s^*}$.

A rewrite strategy s on $\mathcal{A}$ is *normalizing* for $x \in X$ if the set $s^* \cdot x$ contains a normal form of $\mathcal{A}$. When s is normalizing for all $x \in X$, s is *normalizing*.

Given two rewrite strategies s and s' on $\mathcal{A}$ and $x \in X$, s *dominates* s' for x if

- either s is normalizing for x and s' is not normalizing for x;
- or both s and s' are normalizing for x and $\# \lfloor s^* \cdot x \rfloor \leqslant \# \lfloor s'^* \cdot x \rfloor$.

When s dominates s' for all $x \in X$, s *dominates* s'.

Example

Let the ARS $\mathcal{A} := (\mathbb{N} \setminus \{0\}, \Rightarrow)$ where $\Rightarrow$ is defined by $n \Rightarrow k$ for any $n, k \in \mathbb{N}$ such that k is a proper divisor of n. The only normal form of $\mathcal{A}$ is 1.

Let s be the strategy on $\mathcal{A}$ such that for any $n \in \mathbb{N}$, $s \cdot n$ is the greatest proper divisor of n. For instance, $s \cdot 123 = 41$ since $123 = 3 \times 41$. Since $\mathcal{A}$ is normalizing, s is normalizing.

Example

Let the ARS $\mathcal{A} := (\mathbb{N}, \Rightarrow)$ where $\Rightarrow$ is defined by $n \Rightarrow k$ for any $n, k \in \mathbb{N}$ such that n is even and $k \geqslant n$. The normal forms of $\mathcal{A}$ are the odd natural numbers. Moreover, since for any $n \in \mathbb{N}$ such that n is even, we have that $n \Rightarrow n + 1$ and that $n + 1$ is odd, so a normal form, $\mathcal{A}$ is normalizing.

- Let s_1 be the strategy on $\mathcal{A}$ such that for any $n \in \mathbb{N}$ such that n is even, $s_1 \cdot n := 2n$. This strategy is not normalizing.
- Let s_2 be the strategy on $\mathcal{A}$ such that for any $n \in \mathbb{N}$ such that n is even, $s_2 \cdot n := n + 1$. This strategy is normalizing.
- Let s_3 be the strategy on $\mathcal{A}$ such that for any $n \in \mathbb{N}$, if n is divisible by 4, then $s_3 \cdot n := n + 1$, and if n is divisible by 2 but not by 4, $s_3 \cdot n := 2n$. This strategy is normalizing.

The strategy s_1 is dominated by s_2, and s_3 is dominated by s_2.

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS and $\preccurlyeq$ be a total order relation on the rewrite rules of $\mathcal{T}$.

- The *left* (resp. *right*) *$\preccurlyeq$-innermost rewrite strategy* is the rewrite strategy $s_{\mathrm{i}}^{\preccurlyeq}$ such that for any $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, if $\mathfrak{t}$ is not a normal form of $\mathcal{T}$, then $s_{\mathrm{i}}^{\preccurlyeq} \cdot \mathfrak{t} := \mathfrak{t}'$ where $\mathfrak{t}'$ is obtained from $\mathfrak{t}$ by modifying the leftmost (resp. rightmost) factor such that its root has **no proper descendent** on which a one-step rewrite can be made.

  This rewrite strategy is analogous to the **call-by-value** evaluation strategy of programming languages.

- The *left* (resp. *right*) *$\preccurlyeq$-outermost rewrite strategy* is the rewrite strategy $s_{\mathrm{o}}^{\preccurlyeq}$ such that for any $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$, if $\mathfrak{t}$ is not a normal form of $\mathcal{T}$, then $s_{\mathrm{o}}^{\preccurlyeq} \cdot \mathfrak{t} := \mathfrak{t}'$ where $\mathfrak{t}'$ is obtained from $\mathfrak{t}$ by modifying the leftmost (resp. rightmost) factor such that its root has **no proper ancestor** on which a one-step rewrite can be made.

  This rewrite strategy is analogous to the **call-by-name** evaluation strategy of programming languages.

If **several rewrite rules** of $\mathcal{T}$ can be considered at a given position, the **smallest w.r.t. $\preccurlyeq$** is considered.

Exercise ●●●○○

Provide a description of the rewrite strategies $s_{\mathrm{i}}^{\preccurlyeq}$ and $s_{\mathrm{o}}^{\preccurlyeq}$ by using rewrite positions and properties on these (like lexicographic order).

Example

Let the TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathbb{N}, \to)$ such that $r_1 := \mathsf{c}_2 1 \mathsf{c}_1 \to 1$, $r_2 := \mathsf{c}_2 12 \to \mathsf{c}_0$, and $r_3 := \mathsf{c}_2 12 \to \mathsf{c}_2 1 \mathsf{c}_0$. Let the total order relation $\leqslant$ on the rewrite rules of $\mathcal{T}$ such that $r_1 \leqslant r_2 \leqslant r_3$.
Let

$$\mathfrak{t} := \mathsf{c}_3 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_2 2 3 \rfloor \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \rfloor 2 \lfloor \mathsf{c}_2 2 \lfloor \mathsf{c}_1 1 \rfloor \rfloor .$$

- ☐ By the left $\leqslant$-innermost rewrite strategy, $\mathfrak{t}$ is rewritten at position 11 by using r_2.
- ☐ By the right $\leqslant$-innermost rewrite strategy, $\mathfrak{t}$ is rewritten at position 3 by using r_1.
- ☐ By the left $\leqslant$-outermost rewrite strategy, $\mathfrak{t}$ is rewritten at position 1 by using r_2.
- ☐ By the right $\leqslant$-outermost rewrite strategy, $\mathfrak{t}$ is rewritten at position 3 by using r_1.

In general, neither the innermost strategy nor the outermost strategy dominates the other.

For the next examples, let $\mathcal{S}$ be the signature containing four nullary constants a, b, c, and d, two unary constants f and g, and one binary constant h.

We shall also consider only the left variations of these strategies and we need not specify any total order $\leqslant$ on the considered rewrite rules.

Example [s_{i} normalizing but s_{o} not]

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $f \lfloor g1 \rfloor \to f \lfloor g \lfloor g1 \rfloor \rfloor$ and $ga \to a$.
From the $\mathcal{S}, \mathbb{N}$-term $\mathfrak{t} := f \lfloor ga \rfloor$, we have

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{i}}} fa$$

and

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{o}}} f \lfloor g \lfloor ga \rfloor \rfloor \Rightarrow_{s_{\mathrm{o}}} f \lfloor g \lfloor g \lfloor ga \rfloor \rfloor \rfloor \Rightarrow_{s_{\mathrm{o}}} f \lfloor g \lfloor g \lfloor g \lfloor ga \rfloor \rfloor \rfloor \rfloor \Rightarrow_{s_{\mathrm{o}}} \cdots .$$

Hence, for a same $\mathcal{S}, \mathbb{N}$-term, the innermost rewrite strategy on $\mathcal{T}$ leads to a normal form while the outermost rewrite strategy leads to an infinite rewrite sequence.

Example [s_{o} normalizing but s_{i} not]

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $f1 \to a$ and $g1 \to g \lfloor g1 \rfloor$.
From the $\mathcal{S}, \mathbb{N}$-term $\mathfrak{t} := f \lfloor ga \rfloor$, we have

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{i}}} f \lfloor g \lfloor ga \rfloor \rfloor \Rightarrow_{s_{\mathrm{i}}} f \lfloor g \lfloor g \lfloor ga \rfloor \rfloor \rfloor \Rightarrow_{s_{\mathrm{i}}} f \lfloor g \lfloor g \lfloor g \lfloor ga \rfloor \rfloor \rfloor \rfloor \Rightarrow_{s_{\mathrm{i}}} \cdots$$

and

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{o}}} a.$$

Hence, for a same $\mathcal{S}, \mathbb{N}$-term, the outermost rewrite strategy on $\mathcal{T}$ leads to a normal form while the innermost rewrite strategy leads to an infinite rewrite sequence.

Example [s_{i} dominates s_{o}]

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $f1 \to h11$, $haa \to b$, and $c \to a$.
From the $\mathcal{S}, \mathbb{N}$-term $\mathfrak{t} := fc$, we have

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{i}}} fa \Rightarrow_{s_{\mathrm{i}}} haa \Rightarrow_{s_{\mathrm{i}}} b$$

and

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{o}}} hcc \Rightarrow_{s_{\mathrm{o}}} hac \Rightarrow_{s_{\mathrm{o}}} haa \Rightarrow_{s_{\mathrm{o}}} b.$$

Hence s_{i} dominates s_{o} for $\mathfrak{t}$ in $\mathcal{T}$.

Example [s_{o} dominates s_{i}]

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $f1 \to a$, $ga \to b$, $gb \to c$, $gc \to d$.
From the $\mathcal{S}, \mathbb{N}$-term $\mathfrak{t} := f \underline{g \underline{g \underline{g a}}}$, we have

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{i}}} f \underline{g \underline{g b}} \Rightarrow_{s_{\mathrm{i}}} f \underline{g c} \Rightarrow_{s_{\mathrm{i}}} fd \Rightarrow_{s_{\mathrm{i}}} a$$

and

$$\mathfrak{t} \Rightarrow_{s_{\mathrm{o}}} a.$$

Hence s_{o} dominates s_{i} for $\mathfrak{t}$ in $\mathcal{T}$.

11.2. Constructor term rewrite systems

Definition

A *constructor TRS* (*CTRS*) is a quadruple $(\mathbf{C}, \mathsf{F}, \mathsf{V}, \to)$ where $\mathbf{C}$ is a signature, called the *underlying signature of constructors*, F is a signature, called the *underlying signature of functions*, and such that

1. the underlying sets of $\mathbf{C}$ and F are disjoint;
2. the triple $\mathcal{T} := (\mathbf{C} \sqcup \mathsf{F}, \mathsf{V}, \to)$ is a TRS;
3. for any rewrite rule $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{T}$, $\mathfrak{t}$ is a $\mathbf{C} \sqcup \mathsf{F}, \mathsf{V}$-term of the form $\mathfrak{t} = f\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ where $f \in \mathsf{F}\cdot n$, $n \in \mathbb{N}$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T}\cdot\mathbf{C}\cdot\mathsf{V}$.

A CTRS $\mathcal{C} := (\mathbf{C}, \mathsf{F}, \mathsf{V}, \to)$ defines the TRS $(\mathbf{C} \sqcup \mathsf{F}, \mathsf{V}, \to)$, called the *TRS of* $\mathcal{C}$. We use the previous notations and notions in the context of TRSs, here on the TRS of $\mathcal{C}$.

Example

Let $\mathcal{C} := (\mathbf{C}, \mathsf{F}, \mathbb{N}, \to)$ be the CTRS defined as follows. Let the signature $\mathbf{C}$ containing the three nullary constants $\mathbf{true}$, $\mathbf{false}$, and $\mathbf{zero}$, and the unary constant $\mathbf{succ}$. Let the signature F containing the three unary constants not, even, and odd. Let $\to$ be the elementary rewrite relation defined by $\mathsf{not}\,\mathbf{true} \to \mathbf{false}$, $\mathsf{not}\,\mathbf{false} \to \mathbf{true}$, $\mathsf{even}\,\mathbf{zero} \to \mathbf{true}$, $\mathsf{even}\,\llcorner\mathbf{succ}\,\mathbf{zero}\lrcorner \to \mathbf{false}$, $\mathsf{even}\,\llcorner\mathbf{succ}\,\llcorner\mathbf{succ}\,1\lrcorner\lrcorner \to \mathsf{even}\,1$, and $\mathsf{odd}\,1 \to \mathsf{not}\,\llcorner\mathsf{even}\,1\lrcorner$.

The quadruple $\mathcal{C}$ is a CTRS.

CTRSs can be used as a programming framework:

- constructors make the **data** manipulated by the program;
- functions are the **computing units** of the program;
- elementary rewrite relations describe how to compute the **result of applying a function to a sequence of values**.

They form a simplified version of **pattern matching** of **functional programming languages**.

There are some interesting points about a CTRS $\mathcal{C} := (\mathbf{C}, \mathsf{F}, \mathbb{N}, \to)$:

- any critical data (r_1, u, r_2) of $\mathcal{C}$ is such that $u = \epsilon$. In other words, left members of rules of $\mathcal{C}$ can overlap only at the root;
- Any $\mathbf{C}, \mathsf{V}$-term is a **normal form** of $\mathcal{C}$. The converse is false.

Definition

A *recursive program scheme* (*RPS*) is a CTRS $\mathcal{C} := (\mathbf{C}, \mathsf{F}, \mathbb{N}, \to)$ such that

1. for any rewrite rule $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{C}$, $\mathfrak{t} = f\, v_1 \dots v_n$ where $f \in \mathsf{F} \cdot n$, $n \in \mathbb{N}$, $v_1, \dots, v_n \in \mathsf{V}$, and $v_i = v_{i'}$ implies $i = i'$;
2. for any $f \in \mathsf{F} \cdot n$, $n \in \mathbb{N}$, there is exactly one rewrite rule of $\mathcal{C}$ of the form of 1..

Example

The CTRS $\mathcal{C} := (\mathbf{C}, \mathsf{F}, \mathbb{N}, \to)$ such that $\mathbf{C}$ is the signature containing the nullary constant $\mathbf{l}$ and the binary constant $\mathbf{n}$, F is the signature containing a binary constant f and a ternary constant g, and $\to$ is defined by $\mathsf{f}12 \to \mathbf{n}21$ and $\mathsf{g}123 \to \mathbf{n}\,\underline{\mathbf{n}12}\,\underline{\mathbf{n}23}$.

It is immediate that any RPS is orthogonal.

Hence, by Theorem [Confluence of weakly orthogonal TRSs], any RPS is confluent.

Theorem [CTRSs and universality of computation]

For any computable partial function f between two sets X and Y, there exist a CTRS $\mathcal{C}_f := (\mathsf{C}, \mathsf{F}, \mathsf{V}, \to)$ and two coding functions $c_X : X \to \mathfrak{T}\cdot\lfloor \mathsf{C} \sqcup \mathsf{F} \rfloor\cdot\emptyset$ and $c_Y : Y \to \mathfrak{T}\cdot\lfloor \mathsf{C} \sqcup \mathsf{F} \rfloor\cdot\emptyset$ such that, for any $x \in X$,

- if $f\cdot x$ is defined, then $c_X\cdot x$ has $c_Y\cdot\lfloor f\cdot x \rfloor$ as a normal form in $\mathcal{C}_f$;
- otherwise, $c_X\cdot x$ has no normal form in $\mathcal{C}_f$.

Theorem [CTRSs and universality of computation] says that CTRSs form a Turing-complete programming language.

Theorem [Decidability of normalization in RPSs]

The problem of deciding, given an RPS $\mathcal{C} := (\mathsf{C}, \mathsf{F}, \mathsf{V}, \to)$ and a $\lfloor \mathsf{C} \sqcup \mathsf{F} \rfloor, \mathsf{V}$-term $\mathfrak{t}$, whether $\mathfrak{t}$ admits a normal form in $\mathcal{C}$ is decidable.

Theorem [Decidability of normalization in RPSs] appears in [Z. Khasidashvili, Optimal normalization in orthogonal term rewriting systems, 1993].

Since normalization on a given input is undecidable for any Turing-complete programming language, Theorem [Decidability of normalization in RPSs] implies that RPSs are not Turing-complete.

Exercise ●●○○○

Define a CTRS which represents natural numbers (as Peano integers), their addition, multiplication, and exponentiation.

Exercise ●●○○○

1. Define a CTRS which is not an RPS and is not terminating.
2. Define a CTRS which is not an RPS and is is not confluent.
3. Define an RPS which is not terminating.
4. Define an RPS which is not confluent.

Exercise ●●●○○

Define a CTRS able to represent, in particular, the OCaml expression

```
if x = y then e else e'
```

where `x` and `y` are any natural numbers, and `e` and `e'` are any expressions.

11.3. Applicative systems

A signature $\mathcal{S}$ is *applicative* if $\mathcal{S} \cdot 2$ is a singleton and for any $n \in \mathbb{N} \setminus \{0, 2\}$, $\mathcal{S} \cdot n = \emptyset$.

The unique element of arity 2 of an applicative signature is denoted by $\bullet$. This constant is the *application constant*.

For any set C, let $[C]$ be the applicative signature such that $[C] \cdot 0 = C$.

Example

The applicative signature $[\{a, b, c\}]$ is such that $[\{a, b, c\}] \cdot 0 = \{a, b, c\}$, $[\{a, b, c\}] \cdot 1 = \emptyset$, $[\{a, b, c\}] \cdot 2 = \{\bullet\}$, and $[\{a, b, c\}] \cdot n = \emptyset$ for any $n \geqslant 3$.

Given a signature $\mathcal{S}$ and a set of variables V, an $\mathcal{S}, \mathsf{V}$-term is *applicative* if $\mathcal{S}$ is applicative.

Given an applicative $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, the *concise notation* of $\mathfrak{t}$ is obtained by deleting all symbols $\bullet$ from the applicative notation of $\mathfrak{t}$.

Example

Let us consider the previous applicative signature. The labeled $\mathcal{S}$-term $\bullet \lfloor \bullet \lfloor \bullet a \lfloor \bullet b a \rfloor \rfloor 1 \rfloor \lfloor \bullet b \lfloor \bullet 2 1 \rfloor \rfloor$ admits $a \lfloor b a \rfloor 1 \lfloor b \lfloor 2 1 \rfloor \rfloor$ as concise notation.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *curried* version of $\mathcal{S}$ is the **applicative signature** $\mathrm{cur} \cdot \mathcal{S}$ such that $\mathrm{cur} \cdot \mathcal{S}(0) = \bigsqcup_{n \in \mathbb{N}} \mathcal{S}(n)$.

The *curried* version of an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ is the $\mathrm{cur} \cdot \mathcal{S}, \mathsf{V}$**-term** $\mathrm{cur} \cdot \mathfrak{t}$ defined by

$$\mathrm{cur} \cdot \mathfrak{t} := \begin{cases} v & \text{if } \mathfrak{t} = v \text{ and } v \in \mathsf{V}, \\ \bullet \underbracket{\ldots \underbracket{\bullet\, c \underbracket{\mathrm{cur} \cdot \mathfrak{t}_1}} \ldots} \underbracket{\mathrm{cur} \cdot \mathfrak{t}_n} & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \ldots \mathfrak{t}_n, c \in \mathcal{S}(n), n \in \mathbb{N}, \mathfrak{t}_1, \ldots, \mathfrak{t}_n \in \mathfrak{T}(\mathcal{S})(\mathsf{V}). \end{cases}$$

Example

Let the labeled $\mathcal{S}_{\mathbb{N}^2}$-term

$$\mathfrak{t} := \mathsf{c}_3 1 \underbracket{\mathsf{c}_2 \mathsf{c}_0 \underbracket{\mathsf{c}_1 2}} \underbracket{\mathsf{c}_3 2 \mathsf{c}_0 3}.$$

The curried version of $\mathfrak{t}$ is the labeled $\mathrm{cur} \cdot \mathcal{S}_{\mathbb{N}^2}$-term

$$\mathrm{cur} \cdot \mathfrak{t} = \bullet \underbracket{\bullet \underbracket{\bullet \mathsf{c}_3 1} \underbracket{\bullet \underbracket{\mathsf{c}_2 \mathsf{c}_0} \underbracket{\mathsf{c}_1 2}}} \underbracket{\bullet \underbracket{\bullet \underbracket{\bullet \mathsf{c}_3 2} \mathsf{c}_0} 3}.$$

Note that the **applicative notation** of an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ and the **concise notation** of $\mathrm{cur} \cdot \mathfrak{t}$ are the same strings.

A TRS $\mathcal{T}$ is *applicative* if its underlying signature is applicative.

Given a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$, the *curried* version of $\mathcal{T}$ is the TRS $\mathrm{cur}{\cdot}\mathcal{T} := (\mathrm{cur}{\cdot}\mathcal{S}, \mathsf{V}, \mathrm{cur}{\cdot} \to)$ such that $\mathrm{cur}{\cdot} \to$ is the elementary rewrite relation defined by $(\mathrm{cur}{\cdot}\mathfrak{t}, \mathrm{cur}{\cdot}\mathfrak{t}') \in \mathrm{cur}{\cdot} \to$ if $\mathfrak{t}$ and $\mathfrak{t}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t} \to \mathfrak{t}'$.

Example

Let the TRS $\mathsf{NatAdd} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing one nullary constant z, one unary constant s, and one binary constant a, and such that $a1z \to 1$ and $a1\llcorner s2 \lrcorner \to s\llcorner a12 \lrcorner$.

In $\mathrm{cur}{\cdot}\mathsf{NatAdd}$, the underlying applicative signature contains z, s, and a as three nullary constants. Moreover, the elementary rewrite relation of this TRS is defined by

$$\bullet\llcorner\bullet a1\lrcorner z \ \ \mathrm{cur}{\cdot} \to \ \ 1$$

and

$$\bullet\llcorner\bullet a1\lrcorner\llcorner\bullet s2\lrcorner \ \ \mathrm{cur}{\cdot} \to \ \ \bullet s\llcorner\bullet\llcorner\bullet a1\lrcorner 2\lrcorner.$$

By construction, the curried version of a TRS is an applicative TRS.

Theorem [Curried version of TRS and termination preservation]

Let $\mathcal{T}$ be a TRS. If $\mathcal{T}$ is terminating, then $\mathrm{cur}{\cdot}\mathcal{T}$ is terminating.

This result appears in [J. R. Kennaway, J. W. Klop, M. R. Sleep, F. J. de Vries, Transfinite reductions in orthogonal term rewriting systems, 1995].

Theorem [Curried version of TRS and confluence preservation]

Let $\mathcal{T}$ be a TRS. If $\mathcal{T}$ is confluent, then $\mathrm{cur}{\cdot}\mathcal{T}$ is confluent.

This result appears in [S. Kahrs, Confluence of curried term-rewriting systems, 1995].

Term rewriting programming

11.4. Combinatory logic

A *basic combinator* is a triple $b := (c, n_c, \mathfrak{t}_c)$ such that

- c is a symbol, the *constant* of b;
- $n_c \in \mathbb{N} \setminus \{0\}$ is the *order* of b;
- $\mathfrak{t}_c$ is a labeled $\{\bullet\}$-term such that $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}_c \leqslant n_c$, called the *right member* of b.

Definition

A *combinatory logic system* (*CLS*) is an applicative TRS $\mathcal{C} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \to)$ such that for any $c \in \mathcal{S} \cdot 0$, there is a unique basic combinator $(c, n_c, \mathfrak{t}_c)$ such that

$$c\,1 \ldots n_c \to \mathfrak{t}_c,$$

and all rewrite rules of $\mathcal{C}$ are of this form.

Example

Let $\mathcal{S}$ be the applicative signature $[\{a, b\}]$. Let the basic combinators $(a, 4, 11\llcorner 213 \lrcorner 2)$ and $(b, 3, 21\llcorner 33 \lrcorner\llcorner 22 \lrcorner)$, and let us denote by $\mathcal{C}$ the CLS specified by $\mathcal{S}$ and these two basic combinators.

The elementary rewrite relation $\Rightarrow$ of $\mathcal{C}$ satisfies $a1234 \to 11\llcorner 213 \lrcorner 2$ and $b123 \to 21\llcorner 33 \lrcorner\llcorner 22 \lrcorner$.

A *combinator* of a CLS $\mathcal{C}$ having $\mathcal{S}$ as underlying applicative signature is a ground labeled $\mathcal{S}$-term.

Here are some basic combinators:

- *Idiot Bird*: $(\mathbf{I}, \mathbf{1}, \mathbf{1})$;
- *Mockingbird*: $(\mathbf{M}, 1, 11)$;
- *Kestrel*: $(\mathbf{K}, 2, 1)$;
- *Thrush*: $(\mathbf{T}, 2, 21)$;
- *Warbler*: $(\mathbf{W}, 2, 112)$;
- *Lark*: $(\mathbf{L}, 2, 1\lfloor 22 \rfloor)$;
- *Owl*: $(\mathbf{O}, 2, 2\lfloor 12 \rfloor)$;
- *Turing Bird*: $(\mathbf{U}, 2, 2\lfloor 112 \rfloor)$;
- *Cardinal*: $(\mathbf{C}, 3, 132)$;
- *Vireo*: $(\mathbf{V}, 3, 312)$;
- *Bluebird*: $(\mathbf{B}, 3, 1\lfloor 23 \rfloor)$;
- *Starling*: $(\mathbf{S}, 3, 13\lfloor 23 \rfloor)$.

Most of these appear in [R. Smullyan, To Mock a Mockingbird, 1985].

Some of these are very concrete:

- The Idiot Bird, satisfying for any x, $\mathbf{I}x \to x$, is the **identity function**;
- The Kestrel, satisfying for any α and x, $\mathbf{K}\alpha x \to \alpha$, allows us to build constant functions. Indeed, $\mathbf{K}\alpha$ is the function sending any x to α;
- The Thrush, satisfying for any f and x, $\mathbf{T}xf \to fx$ is the reverse application function. In OCaml, this function is denoted by `(|>)`;
- The Bluebird, satisfying for any f_1, f_2, and x, $\mathbf{B}f_1f_2x \to f_1\lfloor f_2x \rfloor$ allows us to compose functions. Indeed, $\mathbf{B}f_1f_2$ is the function sending any x to the application of f_1 on the application of f_2 on x.

Proposition [Confluence of CLSs]

Any CLS is confluent.

This is a consequence of the fact that any CLS is orthogonal and Theorem [Confluence of weakly orthogonal TRSs]. Indeed, the leftmost symbol of any left-hand side of a rewrite rule of a CLS is a unique constant of arity 0 of its underlying signature.

Nevertheless, CLSs are in general not terminating.

Example

Let $\mathcal{C}$ be the CLS on the two basic combinators $\mathbf{I}$ and $\mathbf{S}$.
Let $\mathfrak{t} := \mathbf{S}\llcorner\mathbf{SII}\lrcorner\mathbf{I}$.
Observe that for any variable v,

$$\mathfrak{t}v = \mathbf{S}\llcorner\mathbf{SII}\lrcorner\mathbf{I}v \Rightarrow \mathbf{SII}v\llcorner\mathbf{I}v\lrcorner \Rightarrow \mathbf{I}v\llcorner\mathbf{I}v\lrcorner\llcorner\mathbf{I}v\lrcorner \Rightarrow v\llcorner\mathbf{I}v\lrcorner\llcorner\mathbf{I}v\lrcorner \Rightarrow vv\llcorner\mathbf{I}v\lrcorner \Rightarrow vvv.$$

As a consequence,

$$\mathfrak{tt} \Rightarrow^* \mathfrak{ttt} \Rightarrow^* \mathfrak{tttt} \Rightarrow^* \cdots$$

is an infinite rewrite sequence in $\mathcal{C}$.

Let $\mathcal{C} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \to)$ be a CLS.

Any labeled $\mathcal{S}$-term $\mathfrak{t}$ is seen as a **function**: given a sequence $\mathfrak{t}_1, \dots, \mathfrak{t}_n$, $n \in \mathbb{N}$, of labeled $\mathcal{S}$-terms, the iterated application $\mathfrak{t}\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ is the **application** of $\mathfrak{t}$ to the **arguments** $\mathfrak{t}_1$, ..., $\mathfrak{t}_n$.

Let $\equiv_{\mathrm{e}}$ be the *extensional relation*, defined as the smallest equivalence relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \lfloor \mathbb{N} \setminus \{0\} \rfloor$ such that, for any labeled $\mathcal{S}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$,

- if $\mathfrak{t} \equiv \mathfrak{t}'$, then $\mathfrak{t} \equiv_{\mathrm{e}} \mathfrak{t}'$;
- if $\mathfrak{t} v \equiv_{\mathrm{e}} \mathfrak{t}' v$, where v is any variable having no occurrence in $\mathfrak{t}$ and $\mathfrak{t}'$, then $\mathfrak{t} \equiv_{\mathrm{e}} \mathfrak{t}'$.

Intuitively, when $\mathfrak{t} \equiv_{\mathrm{e}} \mathfrak{t}'$, the labeled $\mathcal{S}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, seen as functions, have the same behavior on generic arguments.

Example

Let $\mathcal{C}$ be the CLS on the three basic combinators $\mathbf{I}$, $\mathbf{K}$, and $\mathbf{S}$.
Let $\mathfrak{t} := \mathbf{SK}$ and $\mathfrak{t}' := \mathbf{KI}$. Since

$$\mathfrak{t}12 = \mathbf{SK}12 \Rightarrow \mathbf{K}2\lfloor 12 \rfloor \Rightarrow 2$$

and

$$\mathfrak{t}'12 = \mathbf{KI}12 \Rightarrow \mathbf{I}2 \Rightarrow 2,$$

we have $\mathfrak{t} \equiv_{\mathrm{e}} \mathfrak{t}'$.

Let $\mathcal{C} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \rightarrow)$ be a CLS.

Let $\mathfrak{s}$ be a labeled $\{\bullet\}$-term such that $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{s} = n$, $n \geqslant 1$. When there is a combinator $\mathfrak{t}$ of $\mathcal{C}$ such that

$$\mathfrak{t}\, 1 \ldots n \Rightarrow^* \mathfrak{s},$$

the combinator $\mathfrak{t}$ *computes* $\mathfrak{s}$, and $\mathfrak{s}$ is *computable* in $\mathcal{C}$.

Example

Let $\mathcal{C}$ be the CLS on the two basic combinators $\mathbf{K}$ and $\mathbf{B}$.

Let the labeled $\{\bullet\}$-term $\mathfrak{s} := 12\lfloor 345 \rfloor$. The combinator $\mathfrak{t} := \mathbf{KBB}\lfloor \mathbf{BBB} \rfloor$ computes $\mathfrak{s}$. Indeed,

$$\mathfrak{t}12345 = \mathbf{KBB}\lfloor \mathbf{BBB} \rfloor 12345 \Rightarrow \mathbf{B}\lfloor \mathbf{BBB} \rfloor 12345 \Rightarrow \lfloor \mathbf{BBB} \rfloor \lfloor 12 \rfloor 345 \Rightarrow \mathbf{B}\lfloor \mathbf{B}\lfloor 12 \rfloor \rfloor 345 \Rightarrow \mathbf{B}\lfloor 12 \rfloor \lfloor 34 \rfloor 5 \Rightarrow 12\lfloor 345 \rfloor = \mathfrak{s}.$$

Definition

A CLS $\mathcal{C}$ is *combinatory complete* if all labeled $\{\bullet\}$-terms are computable in $\mathcal{C}$.

Theorem [Turing completeness of combinatory complete CLSs]

Let $\mathcal{C} := (\mathcal{S}, \mathbb{N} \setminus \{0\}, \rightarrow)$ be a combinatory complete CLS. There exists a translation function $\mathfrak{t} \mapsto \langle \mathfrak{t} \rangle$ from λ-terms to labeled $\mathcal{S}$-terms such that, for any λ-terms $\mathfrak{t}$ and $\mathfrak{t}'$, if $\mathfrak{t}$ reduces to $\mathfrak{t}'$ in the λ-calculus, then $\langle \mathfrak{t} \rangle \Rightarrow^* \langle \mathfrak{t}' \rangle$.

Theorem [Combinatory completeness of the CLS on $\mathbf{K}$ and $\mathbf{S}$]

The CLS on the two basic combinators $\mathbf{K}$ and $\mathbf{S}$ is combinatory complete.

This result appears in [H. B. Curry, R. Feys, Combinatory Logic, Vol. I, 1958]. See also [M. Schönfinkel, Über die Bausteine der mathematischen Logik, 1924].

There are other known combinatory complete CLSs. For instance, the CLS on the four basic combinators $\mathbf{B}$, $\mathbf{C}$, $\mathbf{K}$, and $\mathbf{W}$ is one of these [H. B. Curry, Grundlagen der kombinatorischen Logik, 1930].

Theorem [Minimal order of basic combinators of complete CLSs]

Any combinatory complete CLS has at least one basic combinator of order 3 or more.

This result comes from [R. Legrand, A Basis Result in Combinatory Logic, 1988].