Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

Given $Y \subseteq X$, the *closed sub-ARS of $\mathcal{A}$ generated by Y* is the smallest closed sub-ARS w.r.t. inclusion of $\mathcal{A}$ such that underlying set contains Y.

Proposition [Closed sub-ARS generated by a set]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS and Y be a subset of X. The closed sub-ARS of $\mathcal{A}$ generated by Y is the ARS $(Y', \Rightarrow')$ where $Y' := \bigcup_{y \in Y} y^{\Rightarrow^*}$ and $\Rightarrow' := \Rightarrow \cap Y'^2$.

Exercise ●●○○○

Prove the previous proposition.

Example

The closed sub-ARS of Grid generated by $\{(-3, 4), (2, -1)\}$ is the ARS $(X, \Rightarrow)$ such that $X = \left\{(i, j) \in \mathbb{Z}^2 : (i \geqslant -3 \text{ and } j \geqslant 4) \text{ or } (i \geqslant 2 \text{ and } j \geqslant -1)\right\}$, and $\Rightarrow$ is the restriction of the rewrite relation of Grid to X^2.

Exercise ●○○○○○

Let S be a set and $\mathsf{Sets}_S := (\mathcal{P} \cdot S, \Rightarrow)$ be the ARS such that $Z \Rightarrow Z'$ if $Z' = Z \sqcup \{s\}$ for an $s \in S \setminus Z$.

For instance, in $\mathsf{Sets}_{\mathbb{N}}$,

$$\{1, 4\} \Rightarrow \{1, 4, 5\} \Rightarrow \{1, 3, 4, 5\} \Rightarrow \{1, 3, 4, 5, 9\}.$$

1. Rephrase the definition of Sets_S by using a successor function.
2. Describe a necessary and sufficient condition for the property of Sets_S to be finitely branching.
3. Describe the dual of Sets_S by giving its successor function.
4. Describe the future function of Sets_S.
5. Prove that Sets_S is acyclic.
6. Define a sub-ARS of Sets_S which is not an induced sub-ARS of Sets_S.
7. Define an induced sub-ARS of Sets_S which is not a closed sub-ARS of Sets_S.
8. Define a closed sub-ARS of Sets_S which is not Sets_S itself.

Exercise ●●●○○

Let, for any $n \in \mathbb{N}$, $\mathfrak{S}_n$ be the set of permutations of size n. For instance,

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Let $\mathsf{Permutations}_n := (\mathfrak{S}_n, \Rightarrow)$ be the ARS such that $\sigma \Rightarrow \sigma'$ if $\sigma' = \sigma \circ s_i$ where $i \in [n-1]$, s_i is an elementary transposition of $\mathfrak{S}_n$, and $\sigma \cdot i < \sigma \cdot \llcorner i + 1 \lrcorner$.

For instance, in $\mathsf{Permutations}_6$, we have $452361 \Rightarrow 452631$ since $452631 = 452361 \circ s_4$ and $452361 \cdot 4 = 3 < 6 = 452361 \cdot 5$. Besides, in $\mathsf{Permutations}_7$, we have

$$3714652 \Rightarrow 7314652 \Rightarrow 7316452 \Rightarrow 7316542.$$

1. Rephrase the definition of $\mathsf{Permutations}_n$ by using a successor function.
2. Prove that all rewrite sequences in $\mathsf{Permutations}_n$ are finite.
3. Describe the future function of $\mathsf{Permutations}_n$. In other terms, provide a necessary and sufficient combinatorial criterion on $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_n$ for the property $\sigma \Rightarrow^* \sigma'$, without having to exhibit a rewrite sequence starting from σ and ending at σ'.

/ Abstract rewrite systems

3.3. Termination

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

An $x \in X$ is a *normal form* of $\mathcal{A}$ if $x^{\Rightarrow} = \emptyset$.

A rewrite sequence u in $\mathcal{A}$ is *normalizing* if u is **finite** and **ends at a normal form** of $\mathcal{A}$.

In $\mathcal{A}$, $x \in X$ is

- *non-normalizing* if $x^{\Rightarrow^*}$ contains no normal form of $\mathcal{A}$;
- *normalizing* if there exists a normalizing rewrite sequence in $\mathcal{A}$ which starts from x;
- *terminating* if all rewrite sequences starting from x in $\mathcal{A}$ are finite.

Note that in $\mathcal{A}$,

- the set of **normal forms** is a subset of the set of **terminating** elements;
- the set of **terminating** elements is a subset of the set of **normalizing** elements;
- the set of **non-normalizing** elements is disjoint from the set of **normalizing** elements.

When all elements of X are normalizing (resp. terminating), $\mathcal{A}$ is *normalizing* (resp. *terminating*).

Example

The only normal form of Pred is 0. Since for any $n \in \mathbb{N}$, any rewrite sequence starting from n is of the form $n, n-1, \dots, m$ with $0 \leqslant m \leqslant n$, n is terminating. Therefore, Pred is terminating.

Example

Let the ARS $\mathsf{EvenOdd} := (\mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}, \Rightarrow)$ such that, for any $n \in \mathbb{N}$, $n \Rightarrow n+1$, $n \Rightarrow \mathrm{e}$ if n is even, and $n \Rightarrow \mathrm{o}$ if n is odd. The only normal forms of $\mathsf{EvenOdd}$ are e and o. Since for any $n \in \mathbb{N}$, either $n\mathrm{e}$ or $n\mathrm{o}$ is a normalizing rewrite sequence in $\mathsf{EvenOdd}$ starting from n, n is normalizing. Moreover, since $n, n + 1, \dots$ is an infinite rewrite sequence in $\mathsf{EvenOdd}$, n is not terminating. Therefore, $\mathsf{EvenOdd}$ is normalizing and not terminating.

Example

Let $\mathcal{A} := (\{a, b, c\}, \Rightarrow)$ be the ARS such that $a \Rightarrow b$, $a \Rightarrow c$, and $b \Rightarrow b$. Since $c^{\Rightarrow} = \emptyset$, c is a normal form of $\mathcal{A}$. Besides, since ac is a normalizing rewrite sequence in $\mathcal{A}$, a is normalizing. Moreover, since ab^{ω} is an infinite rewrite sequence in $\mathcal{A}$ (b^{ω} denotes the infinite sequence made of b), a is not terminating. Observe that b is not normalizing since the only rewrite sequences starting from b in $\mathcal{A}$ are of the form b^{n}, $n \in \mathbb{N}$, or b^{ω}. Therefore, $\mathcal{A}$ is not normalizing.

/ Abstract rewrite systems

3.4. Confluence

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

For any $x, x' \in X$, let

$$x \Downarrow x' := x^{\Rightarrow^*} \cap x'^{\Rightarrow^*}.$$

When $x \Downarrow x' \neq \emptyset$, x and x' are *joinable* in $\mathcal{A}$.

Observe that x and x' are joinable in $\mathcal{A}$ iff x and x' have a common element in their futures.

Let $\overline{\Downarrow} := \Rightarrow^* \circ \Leftarrow^*$ be the *joinability relation* of $\mathcal{A}$.

Proposition [Joinability relation]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. For any $x, x' \in X$, x and x' are joinable in $\mathcal{A}$ iff $x \, \overline{\Downarrow} \, x'$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

For any $x, x' \in X$, let

$$x \Uparrow x' := x^{\Leftarrow^*} \cap x'^{\Leftarrow^*}.$$

When $x \Uparrow x' \neq \emptyset$, x and x' are *meetable* in $\mathcal{A}$.

Observe that x and x' are meetable in $\mathcal{A}$ iff x and x' have **a common element in their pasts**.

Let $\bar{\Uparrow} := \Leftarrow^* \circ \Rightarrow^*$ be the *meetability relation* of $\mathcal{A}$.

Proposition [Meetability relation]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. For any $x, x' \in X$, x and x' are meetable in $\mathcal{A}$ iff $x \mathrel{\bar{\Uparrow}} x'$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *confluent* if $\bar{\Uparrow} \subseteq \bar{\Downarrow}$.

Proposition [Confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow^* y$ and $x \Rightarrow^* y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ has the *Church-Rosser property* if $\equiv \subseteq \overline{\Downarrow}$.

Proposition [Church-Rosser property]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ has the Church-Rosser property;
2. for any $y, y' \in X$, if $y \equiv y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *semi-confluent* if $\Leftarrow \circ \Rightarrow^* \subseteq \overline{\Downarrow}$.

Proposition [Semi-confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is semi-confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow y$ and $x \Rightarrow^* y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

Theorem [Semi-confluence, confluence, and Church-Rosser property]

Let $\mathcal{A}$ be an ARS. The following three properties are equivalent:

(1) $\mathcal{A}$ is semi-confluent;

(2) $\mathcal{A}$ is confluent;

(3) $\mathcal{A}$ has the Church-Rosser property.

Proof. Assume that $\mathcal{A} = (X, \Rightarrow)$. By using Propositions [Church-Rosser property], [Confluence], and [Semi-confluence], we have that (3) implies (2), and that (2) implies (1).

Conversely, assume that (1) holds. Let $y, y' \in X$ such that $y \equiv y'$. There exists $k \geqslant 0$ and $y_0, y_1, \dots, y_k \in X$ such that $y = y_0 \Leftrightarrow y_1 \Leftrightarrow \cdots \Leftrightarrow y_k = y'$. We prove $y \mathrel{\bar{\Downarrow}} y'$ by induction on k. If $k = 0$, $y = y'$ and the property holds immediately. Otherwise, $k \geqslant 1$ and, by induction hypothesis, $y \mathrel{\bar{\Downarrow}} y_{k-1}$. Thus, there exists $z \in X$ such that $y \Rightarrow^* z$ and $y_{k-1} \Rightarrow^* z$. We have now two cases:

- ☐ If $y_{k-1} \Rightarrow y'$, then, as we have both $y_{k-1} \Rightarrow^* z$ and $y_{k-1} \Rightarrow y'$, by Proposition [Semi-confluence], $z \mathrel{\bar{\Downarrow}} y'$. Since $y \Rightarrow^* z$, this implies $y \mathrel{\bar{\Downarrow}} y'$.
- ☐ Otherwise, $y' \Rightarrow y_{k-1}$. We have $y' \Rightarrow^* z$ and $y \Rightarrow^* z$ so that $y \mathrel{\bar{\Downarrow}} y'$.

This shows that $y \equiv y'$ implies $y \mathrel{\bar{\Downarrow}} y'$, proving that $\mathcal{A}$ has the Church-Rosser property. Therefore, (1) implies (3).

Proposition [Finite Church-Rosser property]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS having the Church-Rosser property. For any $y_1, \dots, y_n \in X$, $n \in \mathbb{N}$, if $y_i \equiv y_j$ for all $i, j \in [n]$, then there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n]$.

Proof. We proceed by induction on n. The property holds vacuously when $n = 0$. When $n = 1$, $z = y_1$ satisfies the property. When $n = 2$, this is exactly the property stated by Proposition [Church-Rosser property]. Assume now that $n \geqslant 3$ and $y_i \equiv y_j$ for all $i, j \in [n]$. By induction hypothesis, there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n-1]$. Now, since $z \equiv y_n$, by Proposition [Church-Rosser property], there exists $z' \in X$ such that $z \Rightarrow^* z'$ and $y_n \Rightarrow^* z'$. As $y_i \Rightarrow^* z'$ for any $i \in [n]$, the desired property is established.

Note that this result, by using Proposition [Confluence] and Theorem [Semi-confluence, confluence, and Church-Rosser property] implies an analogous property for confluence, the **finite confluence property**: If $\mathcal{A} := (X, \Rightarrow)$ is a confluent ARS, then for any $x, y_1, \dots, y_n \in X$, $n \in \mathbb{N}$, the property $x \Rightarrow^* y_i$ for all $i \in [n]$ implies that there exists $z \in X$ such that $y_i \Rightarrow^* z$ for all $i \in [n]$.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The ARS $\mathcal{A}$ is *locally confluent* if $\Leftarrow \circ \Rightarrow \subseteq \overline{\Downarrow}$.

Proposition [Local confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. The following two properties are equivalent:

1. $\mathcal{A}$ is locally confluent;
2. for any $x, y, y' \in X$, if $x \Rightarrow y$ and $x \Rightarrow y'$, then there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$.

Exercise ●○○○○

Prove the previous proposition.

In contrast with Theorem [Semi-confluence, confluence, and Church-Rosser property], **a locally confluent ARS is not necessarily confluent**.

Example

Let us consider the ARS EvenOdd defined previously.

For any $x, x' \in \mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}$, $x \, (\Leftarrow \circ \Rightarrow) \, x'$ iff there exists $y \in \mathbb{N} \cup \{\mathrm{e}, \mathrm{o}\}$ such that $x \Leftarrow y$ and $y \Rightarrow x'$.

We have one of the following possibilities:

(1) $x = n + 1$, $y = n$, $x' = \mathrm{e}$, $n \in \mathbb{N}$ even;
(2) $x = \mathrm{e}$, $y = n$, $x' = n + 1$, $n \in \mathbb{N}$ even;
(3) $x = \mathrm{e}$, $y = n$, $x' = \mathrm{e}$, $n \in \mathbb{N}$ even;
(4) $x = n + 1$, $y = n$, $x' = \mathrm{o}$, $n \in \mathbb{N}$ odd;
(5) $x = \mathrm{o}$, $y = n$, $x' = n + 1$, $n \in \mathbb{N}$ odd;
(6) $x = \mathrm{o}$, $y = n$, $x' = \mathrm{o}$, $n \in \mathbb{N}$ odd.

In Cases (1), (2), and (3), we have $\mathrm{e} \in x \Downarrow x'$

In Cases (4), (5), and (6), we have $\mathrm{o} \in x \Downarrow x'$.

In each case, $x \mathrel{\overline{\Downarrow}} x'$ holds so that EvenOdd is **locally confluent**.

Since e and o are not joinable, this ARS is **not confluent**.

/ Abstract rewrite systems

3.5. General properties

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

The set of **normal forms** in the **future** of $x \in X$ in $\mathcal{A}$ is denoted by $x^{\Rightarrow}$.

Proposition [Termination and normal forms]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is terminating, then for any $x \in X$, $x^{\Rightarrow} \neq \emptyset$.

Exercise ●○○○○

Prove the previous proposition.

Proposition [Confluence and normal forms]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is confluent, then for any $x \in X$, $x^{\Rightarrow}$ is empty or is a singleton.

Exercise ●●○○○

Prove the previous proposition.

Proposition [Confluence, normal forms, and convertibility]

Let $\mathcal{A} := (X, \Rightarrow)$ be a confluent ARS. For any $x \in X$, if there exists $y \in [x]_{\equiv}$ such that y is a normal form of $\mathcal{A}$, then y is the unique normal form of $[x]_{\equiv}$ in $\mathcal{A}$ and $y \in x^{\Rightarrow}$.

Proof. Let us first prove that y is the unique normal form of $[x]_{\equiv}$ in $\mathcal{A}$. For this, let $y' \in [x]_{\equiv}$ be a normal form of $\mathcal{A}$. Since $y, y' \in [x]_{\equiv}$, we have $y \equiv y'$. Moreover, as $\mathcal{A}$ is confluent, by Theorem [Semi-confluence, confluence, and Church-Rosser property] and Proposition [Church-Rosser property], there exists $z \in X$ such that $y \Rightarrow^* z$ and $y' \Rightarrow^* z$. Since y and y' are normal forms of $\mathcal{A}$, $y = z = y'$. Hence, $y = y'$ as expected.

Let us finally prove that $y \in x^{\Rightarrow}$. Since $x \equiv y$, by Theorem [Semi-confluence, confluence, and Church-Rosser property] and Proposition [Church-Rosser property], there exists $z \in X$ such that $x \Rightarrow^* z$ and $y \Rightarrow^* z$. As y is a normal form of $\mathcal{A}$, $y = z$. Therefore, we have $x \Rightarrow^* y$ as expected.

Proposition [Existence of a unique normal form and confluence]

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If for any $x \in X$, $x^{\Rightarrow}$ is a singleton, then $\mathcal{A}$ is confluent.

Proof. Let $x, y, y' \in X$ such that $x \Rightarrow^* y$ and $x \Rightarrow^* y'$. By hypothesis, there exists a unique $z \in X$ such that $z \in y^{\Rightarrow}$ (resp. $z' \in X$ such that $z' \in y'^{\Rightarrow}$). Moreover, since $x \Rightarrow^* y$ and $y \Rightarrow^* z$ (resp. $x \Rightarrow^* y'$ and $y' \Rightarrow^* z'$), we have $x \Rightarrow^* z$ (resp. $x \Rightarrow^* z'$). From the hypothesis, $x^{\Rightarrow}$ is a singleton, implying that $z = z'$. By Proposition [Confluence], this implies that $\mathcal{A}$ is confluent.

Exercise ●●○○○

Prove that the statement obtained from the one of the previous proposition by asking that $\# x^{\Rightarrow} \leqslant 1$ instead of $\# x^{\Rightarrow} = 1$ does not implies that $\mathcal{A}$ is confluent.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

When $\mathcal{A}$ is both terminating and confluent, $(X, \Rightarrow)$ is *convergent*.

Note that the properties of termination and confluence are **independent**.

Exercise ●○○○○

Define a non-terminating and non-confluent ARS, a terminating and non-confluent ARS, a non-terminating and confluent ARS, and a convergent ARS.

Theorem (Convergence and quotient by the convertibility relation)

Let $\mathcal{A} := (X, \Rightarrow)$ be a convergent ARS. The set of normal forms of $\mathcal{A}$ is a complete set of representatives of the quotient $X/_{\equiv}$.

Proof. By Propositions [Termination and normal forms] and [Confluence and normal forms], each $\equiv$-equivalence class contains exactly one normal form of $\mathcal{A}$. This implies the statement of the Theorem.

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

An $x \in X$ is *ambiguous* in $\mathcal{A}$ if there exist $y, y' \in x^{\Rightarrow}$ such that $y \neq y'$.

Theorem (Newman's Lemma)

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS. If $\mathcal{A}$ is terminating and locally confluent, then $\mathcal{A}$ is confluent.

Proof. Assume that $x \in X$ is ambiguous in $\mathcal{A}$ so that there exist $z, z' \in x^{\Rightarrow}$ with $z \neq z'$. Hence, we have $x \Rightarrow y \Rightarrow^* z$ and $x \Rightarrow y' \Rightarrow^* z'$ for some $y, y' \in X$. By the fact that $\mathcal{A}$ is locally confluent and by Proposition [Local confluence], there is $t \in X$ such that $y \Rightarrow^* t$ and $y' \Rightarrow^* t$. By Proposition [Termination and normal forms], $t^{\Rightarrow}$ contains a $u \in X$. Since $z \neq z'$, we have $u \neq z$ or $u \neq z'$ (or both). If $u \neq z$, then as $y \Rightarrow^* z$, $y \Rightarrow^* u$, and z and u are normal forms of $\mathcal{A}$, y is ambiguous. Similarly, If $u \neq z'$, y' is ambiguous. This shows that each ambiguous element of X admits at least one ambiguous successor.

This property implies that there is in $\mathcal{A}$ an infinite rewrite sequence consisting of ambiguous elements. Since $\mathcal{A}$ is terminating, $\mathcal{A}$ cannot have any ambiguous element.

Therefore, for any $x \in X$, by Proposition [Termination and normal forms], $x^{\Rightarrow}$ is a singleton. By Proposition [Existence of a unique normal form and confluence], $\mathcal{A}$ is confluent.