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4. Combinatorics of terms

4.1. Terms

Definition

A *graded set* is a pair (X, rk) where X is a set, called the *underlying set*, and $\mathrm{rk} : X \to \mathbb{N}$ is a function, called the *rank function*.

Let $\mathcal{G} := (X, \mathrm{rk})$ be a graded set.

For any $x \in X$, the natural number $\mathrm{rk} \cdot x$ is the *rank* of x in $\mathcal{G}$.

For any $n \in \mathbb{N}$, let

$$\mathcal{G} \cdot n := \{x \in X : \mathrm{rk} \cdot x = n\}.$$

If for any $n \in \mathbb{N}$, $\mathcal{G} \cdot n$ is finite, then $\mathcal{G}$ is *combinatorial*.

A graded set $\mathcal{G}' := \left(X', \mathrm{rk}'\right)$ is a *sub-graded set* of $\mathcal{G}$ if $X' \subset X$ and rk' is the restriction of rk on the domain X'.

Examples

Let the graded set $\mathcal{G} := (\{a, b\}^*, \ell)$. Since there are finitely many words on $\{a, b\}$ of any length $n \in \mathbb{N}$, $\mathcal{G}$ is combinatorial.

Let the graded set $\mathcal{G}' := (\{a, b\}^*, \ell_a)$. Since $\mathcal{G}' \cdot 0 = \{\epsilon, b, bb, bbb, \dots\}$ is an infinite set, $\mathcal{G}'$ is not combinatorial.

A *signature* $\mathcal{S}$ is a graded set whose elements of the underlying set of a signature are called *constants* and rank function is called *arity function*.

Example

Let the graded set $\mathcal{S}_{\mathbb{N}^2} := (\{\mathsf{c}_{i,j} : i, j \in \mathbb{N}\}, \mathrm{ar})$ where for any $\mathsf{c}_{i,j} \in \mathcal{S}_{\mathbb{N}^2}$, $\mathrm{ar} \cdot \mathsf{c}_{i,j} = i$.

In the sequel, to lighten the notations, we shall write simply c_i for $\mathsf{c}_{i,0}$.

A *set of variables* V is a set whose elements are called *variables*.

Example

Let the set of variables $\mathsf{V}_{\mathbb{N}} := \{\mathsf{v}_i : i \in \mathbb{N}\}$.

Let, for any $n \in \mathbb{N}$, V_n be the subset of $\mathsf{V}_{\mathbb{N}}$ containing only the variables v_i such that $i \in [n]$.

We **always implicitly assume** that any set of variables is **disjoint** from the underlying set of any signature.

Definition

Given a signature $\mathcal{S}$ and a set of variables V, an $\mathcal{S}, \mathsf{V}$-*term* is defined recursively to be

- either a variable $v \in \mathsf{V}$;
- or a pair $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ where $c \in \mathcal{S} \cdot n$ for an $n \in \mathbb{N}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The set of $\mathcal{S}, \mathsf{V}$-terms is denoted by $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

If $\mathfrak{t} = (c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$ is an $\mathcal{S}, \mathsf{V}$-term, for any $j \in [n]$, the *j-th subterm* of $\mathfrak{t}$ is the $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t} \cdot j := \mathfrak{t}_j$.

Examples

Here are some $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms:

$$\mathsf{v}_8, \quad (\mathsf{c}_0, ()), \quad \mathfrak{t} := (\mathsf{c}_2, ((\mathsf{c}_2, (\mathsf{v}_2, \mathsf{v}_1)), (\mathsf{c}_1, (\mathsf{v}_2))))$$

Moreover, we have $\mathfrak{t} \cdot 1 = (\mathsf{c}_2, (\mathsf{v}_2, \mathsf{v}_1))$ and $\mathfrak{t} \cdot 2 = (\mathsf{c}_1, (\mathsf{v}_2))$.

We have also $\lfloor \mathfrak{t} \cdot 1 \rfloor \cdot 2 = \mathfrak{t} \cdot 1 \cdot 2 = \mathsf{v}_1$.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

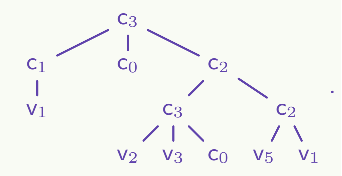
It follows immediately from the definition that $\mathfrak{t}$ is a decorated ordered rooted tree.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := (\mathsf{c}_3, ((\mathsf{c}_1, (\mathsf{v}_1)), (\mathsf{c}_0, ()), (\mathsf{c}_2, ((\mathsf{c}_3, (\mathsf{v}_2, \mathsf{v}_3, (\mathsf{c}_0, ()))), (\mathsf{c}_2, (\mathsf{v}_5, \mathsf{v}_1)))))).$$

This term is represented as the decorated rooted tree



By considering the usual terminology of graphs and trees and by seeing $\mathfrak{t}$ as a tree:

- a *node* of $\mathfrak{t}$ is any vertex of $\mathfrak{t}$. Such nodes are decorated on $\mathsf{V} \sqcup C$;
- an *internal node* of $\mathfrak{t}$ is a node decorated on C;
- a *leaf* of $\mathfrak{t}$ is a node decorated on V.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

The *functional notation* of $\mathfrak{t}$ is obtained by writing $c(\mathfrak{t}_1, \dots, \mathfrak{t}_n)$ instead of $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$.

Example

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of the previous example admits the functional notation

$$\mathsf{c}_3(\mathsf{c}_1(\mathsf{v}_1), \mathsf{c}_0(), \mathsf{c}_2(\mathsf{c}_3(\mathsf{v}_2, \mathsf{v}_3, \mathsf{c}_0()), \mathsf{c}_2(\mathsf{v}_5, \mathsf{v}_1))).$$

The *applicative notation* of $\mathfrak{t}$ is obtained by writing $c\, \mathfrak{t}_1 \dots \mathfrak{t}_n$ instead of $(c, (\mathfrak{t}_1, \dots, \mathfrak{t}_n))$.

Example

The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term of the previous example admits the applicative notation

$$\mathsf{c}_3 \, \underline{\mathsf{c}_1 \mathsf{v}_1} \, \mathsf{c}_0 \, \underline{\mathsf{c}_2 \, \underline{\mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0} \, \underline{\mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1}}.$$

In the sequel, we shall mainly use the applicative notation for $\mathcal{S}, \mathsf{V}$-terms up to some small exceptions.

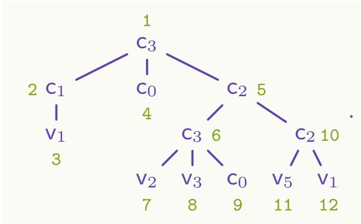
Given a signature $\mathcal{S}$, a set of variables V, and an $\mathcal{S},\mathsf{V}$-term $\mathfrak{t}$, the *preorder traversal* of $\mathfrak{t}$ is defined recursively as follows:

- if $\mathfrak{t} = v$ where $v \in \mathsf{V}$, then the leaf forming $\mathfrak{t}$ is visited;
- otherwise, $\mathfrak{t} = c\,\mathfrak{t}_1 \dots \mathfrak{t}_n$ where $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and $\mathfrak{t}_1$, ..., $\mathfrak{t}_n$ are $\mathcal{S},\mathsf{V}$-terms. The root of $\mathfrak{t}$ is visited first, and then, $\mathfrak{t}_1$, ..., and $\mathfrak{t}_n$ are visited according to their respective preorder traversals.

This procedure induces a total order on the nodes of $\mathfrak{t}$ where the first visited element is the smallest one.

Example

Here is the $\mathcal{S}_{\mathbb{N}^2},\mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ of the previous example with the indices of its nodes according to their order of appearance in the preorder traversal of $\mathfrak{t}$:



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Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term.

- ☐ The *word of* $\mathfrak{t}$ is the word $\mathrm{w} \cdot \mathfrak{t}$ on $\mathrm{V} \cup C$ obtained by reading from left to right the symbols of the applicative notation of $\mathfrak{t}$.
- ☐ The *variable word* $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the subword of $\mathrm{w} \cdot \mathfrak{t}$ made of the letters of V.
- ☐ The *constant word* $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the subword of $\mathrm{w} \cdot \mathfrak{t}$ made of the letters of C.
- ☐ The *variable set* $\mathrm{Vars} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the set $\{\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \cdot i : i \in [\ell \cdot \llcorner \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \lrcorner]\}$.

Example

Let $\mathfrak{t}$ be the $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term of the previous example. Since $\mathsf{c}_3 \llcorner \mathsf{c}_1 \mathsf{v}_1 \lrcorner \mathsf{c}_0 \llcorner \mathsf{c}_2 \llcorner \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \lrcorner \llcorner \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1 \lrcorner \lrcorner$, we have

- ☐ $\mathrm{w} \cdot \mathfrak{t} = \mathsf{c}_3 \mathsf{c}_1 \mathsf{v}_1 \mathsf{c}_0 \mathsf{c}_2 \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1$;
- ☐ $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3 \mathsf{v}_5 \mathsf{v}_1$;
- ☐ $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t} = \mathsf{c}_3 \mathsf{c}_1 \mathsf{c}_0 \mathsf{c}_2 \mathsf{c}_3 \mathsf{c}_0 \mathsf{c}_2$;
- ☐ $\mathrm{Vars} \cdot \mathfrak{t} = \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_5\}$.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

When

- $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} = \epsilon$, $\mathfrak{t}$ is *ground*;
- $\ell_{\mathsf{v}} \cdot \lfloor \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \rfloor \leqslant 1$ for all $\mathsf{v} \in \mathsf{V}$, $\mathfrak{t}$ is *linear*.

Examples

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_3 \mathsf{c}_0 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{c}_0 \rfloor \mathsf{c}_0$ is ground (and thus, also linear);
- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{v}_4 \mathsf{v}_3 \rfloor$ is not ground and is linear;
- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathsf{c}_2 \mathsf{v}_3 \mathsf{v}_3$ is neither ground nor linear.

Exercise ○○○○○

Give a necessary and sufficient condition on a signature $\mathcal{S}$ for the existence of ground $\mathcal{S}, \mathsf{V}$-terms, where V is any set of variables.

Exercise ○○○○○

Give an example of a signature $\mathcal{S}$ so that the set of linear $\mathcal{S}, \{\mathsf{v}\}$-terms is infinite, where v is a variable.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term.

Let the following **rank functions** on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$:

- the *length* $\ell \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w} \cdot \mathfrak{t}$;
- the *variable length* $\ell_{\mathrm{var}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t}$.

  Let also $\ell_v \cdot \mathfrak{t} := \ell_v \cdot \lfloor \mathrm{w}_{\mathrm{var}} \cdot \mathfrak{t} \rfloor$ be the number of occurrences of the variable $v \in \mathsf{V}$ in $\mathfrak{t}$;
- the *constant length* $\ell_{\mathrm{cns}} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the length of $\mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t}$.

  Let also $\ell_c \cdot \mathfrak{t} := \ell_c \cdot \lfloor \mathrm{w}_{\mathrm{cns}} \cdot \mathfrak{t} \rfloor$ be the number of occurrences of the constant $c \in C$ in $\mathfrak{t}$;
- the *height* $\mathrm{ht} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is the maximum number of internal nodes on a downward path starting at the root of $\mathfrak{t}$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := \mathsf{c}_3 \lfloor \mathsf{c}_1 \mathsf{v}_1 \rfloor \mathsf{c}_0 \lfloor \mathsf{c}_2 \lfloor \mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_3 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_2 \mathsf{v}_5 \mathsf{v}_1 \rfloor \rfloor.$$

We have $\ell \cdot \mathfrak{t} = 12$, $\ell_{\mathrm{var}} \cdot \mathfrak{t} = 5$, $\ell_{\mathrm{cns}} \cdot \mathfrak{t} = 7$, and $\mathrm{ht} \cdot \mathfrak{t} = 4$.

Besides, $\mathrm{ht} \cdot \mathsf{v}_1 = 0$ and $\mathrm{ht} \cdot \mathsf{c}_0 = 1$.

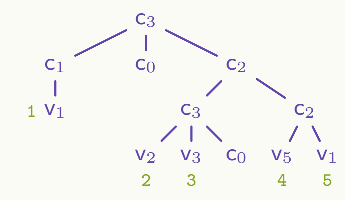
Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term.

Let us consider the following indexation of nodes, leaves, and internal nodes of $\mathfrak{t}$:

- for any $i \in [\ell \cdot \mathfrak{t}]$, the *$i$-th node* of $\mathfrak{t}$ is the i-th node visited according to the preorder traversal of $\mathfrak{t}$;
- for any $i \in [\ell_{\mathrm{var}} \cdot \mathfrak{t}]$, the *$i$-th leaf* of $\mathfrak{t}$ is the i-th leaf visited according to the preorder traversal of $\mathfrak{t}$;
- for any $i \in [\ell_{\mathrm{cns}} \cdot \mathfrak{t}]$, the *$i$-th internal node* of $\mathfrak{t}$ is the i-th internal node visited according to the preorder traversal of $\mathfrak{t}$.

Example

An $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term with the indices of its leaves (left) and the indices of its internal nodes (right):



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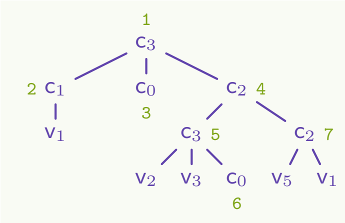

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Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathrm{V}$-term.

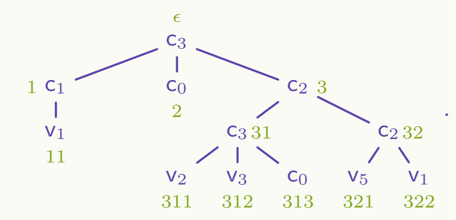
For any word u on $\mathbb{N} \setminus \{0\}$, we define the $\mathcal{S}, \mathrm{V}$-term $\mathfrak{t} \cdot u$ recursively as

$$\mathfrak{t} \cdot u := \begin{cases} \mathfrak{t} & \text{if } u = \epsilon, \\ (\mathfrak{t} \cdot j) \cdot u' & \text{otherwise, where } u = j \, . \, u' \text{ with } j \in \mathbb{N} \setminus \{0\} \text{ and } u' \in (\mathbb{N} \setminus \{0\})^*. \end{cases}$$

The function $u \mapsto \mathfrak{t} \cdot u$ is partial. When $\mathfrak{t} \cdot u$ is defined, the $\mathcal{S}, \mathrm{V}$-term $\mathfrak{t} \cdot u$ is the *u-subterm* of $\mathfrak{t}$ and u is the *position* of the root of $\mathfrak{t} \cdot u$ within $\mathfrak{t}$. The set of positions within $\mathfrak{t}$ is denoted by $\mathrm{P} \cdot \mathfrak{t}$.

Example

Here is an $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-term $\mathfrak{t}$ with the positions of its nodes:



We have

$$\mathrm{P} \cdot \mathfrak{t} = \{\epsilon, 1, 11, 2, 3, 31, 311, 312, 313, 32, 321, 322\}.$$

Definition

Given a signature $\mathcal{S}$, a *labeled $\mathcal{S}$-term* is an $\mathcal{S}, \mathbb{N} \setminus \{0\}$-term.

Let $\mathcal{S}$ be a signature and $\mathfrak{t}$ be a labeled $\mathcal{S}$-term.

The *variable rank* $\mathrm{rk_v} \cdot \mathfrak{t}$ of $\mathfrak{t}$ is 0 if $\mathrm{Vars} \cdot \mathfrak{t} = \emptyset$ and, otherwise, is the maximal element among the variables in $\mathrm{Vars} \cdot \mathfrak{t}$.

Assume that $\mathfrak{t}$ contains n leaves. When

- $\mathrm{w_{var}} \cdot \mathfrak{t} = \llcorner \theta \cdot 1 \lrcorner \dots \llcorner \theta \cdot n \lrcorner$, where $\theta : [n] \to [m]$ is a surjective function for a certain $m \in \mathbb{N}$, $\mathfrak{t}$ is *packed*;
- $\mathrm{w_{var}} \cdot \mathfrak{t} = \llcorner \sigma \cdot 1 \lrcorner \dots \llcorner \sigma \cdot n \lrcorner$, where $\sigma \in \mathfrak{S}_n$, $\mathfrak{t}$ is *standard*;
- $\mathrm{w_{var}} \cdot \mathfrak{t} = 1 \dots n$, $\mathfrak{t}$ is *planar*.

Note that a planar labeled $\mathcal{S}$-term is standard and that a standard labeled $\mathcal{S}$-term is packed.

Examples

- Let $\mathfrak{t} := \mathsf{c}_3 \, 2 \, \llcorner \mathsf{c}_1 \, 1 \lrcorner \, \llcorner \mathsf{c}_2 \, 1 \, 4 \lrcorner$. We have $\mathrm{rk_v} \cdot \mathfrak{t} = 4$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}$ is not packed.
- Let $\mathfrak{t}' := \mathsf{c}_2 \, 2 \, \llcorner \mathsf{c}_2 \, 2 \, 1 \lrcorner$. We have $\mathrm{rk_v} \cdot \mathfrak{t}' = 2$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}'$ is packed but not standard.
- Let $\mathfrak{t}'' := \mathsf{c}_3 \, 2 \, 1 \, 3$. We have $\mathrm{rk_v} \cdot \mathfrak{t}'' = 3$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}''$ is standard but not planar.
- Let $\mathfrak{t}''' := \mathsf{c}_3 \, 1 \, 2 \, 3$. We have $\mathrm{rk_v} \cdot \mathfrak{t}''' = 3$. This labeled $\mathcal{S}_{\mathbb{N}^2}$-term $\mathfrak{t}'''$ is planar.

Exercise ○○○○○

Classify the following sequences of symbols depending on whether they form valid applicative notations of some $\mathcal{S}_{\mathbb{N}^2}, V_{\mathbb{N}}$-terms:

- ☐ v_1 ;
- ☐ c_1 ;
- ☐ c_0 ;
- ☐ $c_2 v_1 v_1$;
- ☐ $c_2 v_2 \llcorner c_1 v_1 \lrcorner$;
- ☐ $c_2 \llcorner c_1 v_1 \lrcorner \llcorner c_0 v_1 \lrcorner$;
- ☐ $c_3 \llcorner v_2 c_0 v_1 \lrcorner v_2 \llcorner c_1 v_4 \lrcorner$;
- ☐ $c_3 \llcorner c_1 c_0 v_1 \lrcorner v_2 \llcorner c_1 v_4 \lrcorner$;
- ☐ $c_3 \llcorner c_2 c_0 v_1 \lrcorner v_2 \llcorner c_1 v_4 \lrcorner$.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}$-term. Give a necessary and sufficient condition for the fact that the word of $\mathfrak{t}$ is a valid applicative notation of an $\mathcal{S}, V$-term.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables.

1. Provide a recursive definition of the height of $\mathcal{S}, V$-terms.
2. Show that for any $\mathcal{S}, V$-term $\mathfrak{t}$, $\mathrm{ht} \cdot \mathfrak{t} \leqslant \ell_{\mathrm{cns}} \cdot \mathfrak{t}$.

Exercise ○○○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and X be a subset of $(\mathbb{N} \setminus \{0\})^*$. Give a necessary and sufficient condition for the fact that there exists $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathrm{P} \cdot \mathfrak{t} = X$.

Exercise ○○○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term. Prove that for any $i \in [\ell \cdot \mathfrak{t}]$, the position u of the i-th node of $\mathfrak{t}$ appears at i-th position among $\mathrm{P} \cdot \mathfrak{t}$ when seen as a sorted list w.r.t. the lexicographic order on $(\mathbb{N} \setminus \{0\})^*$.

Exercise ○○○○○

Given a monoid $(\mathcal{M}, \star, \mathbb{1})$ and a set X, a partial function $\theta : X \to \mathcal{M} \to X$ is a *partial right monoid action* of $\mathcal{M}$ on X if for any $x \in X$, $\theta \cdot x \cdot \mathbb{1} = x$, and for any $x \in X$ and $m_1, m_2 \in \mathcal{M}$, $\theta \cdot \llcorner \theta \cdot x \cdot m_1 \lrcorner \cdot m_2$ is defined iff $\theta \cdot x \cdot \llcorner m_1 \star m_2 \lrcorner$ is defined, and when they are defined, these two elements of X are equal.

Prove that for any signature $\mathcal{S}$ and any set of variables V, the function $\theta : \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \to (\mathbb{N} \setminus \{0\})^* \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ defined for any $\mathfrak{t} \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ and $u \in (\mathbb{N} \setminus \{0\})^*$ by $\theta \cdot \mathfrak{t} \cdot u := \mathfrak{t} \cdot u$ is a partial right monoid action of the free monoid $((\mathbb{N} \setminus \{0\})^*, \centerdot, \epsilon)$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

Exercise ○○○○○

Let, for any $n \in \mathbb{N}$, X_n be the set of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is ground, any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \dots\}$, and $\ell_{\mathsf{c}_0} \cdot \mathfrak{t} = n$.
Let, for any $n \in \mathbb{N}$, X'_n be the set of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}$ such that $\mathfrak{t}$ is planar, any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2, \dots\}$, and $\mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t} = n$.

Describe a bijection $\theta : X_n \to X'_n$.

Exercise ○○○○○

Give a necessary and sufficient condition on a signature $\mathcal{S}$ for the fact that the set of packed labeled $\mathcal{S}$-terms of variable rank n is finite for any $n \in \mathbb{N}$.

Exercise ○○○○○

Provide necessary and sufficient conditions on a signature $\mathcal{S}$ and a set of variables V so that

1. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell)$ is combinatorial;
2. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell_{\mathrm{var}})$ is combinatorial;
3. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \ell_{\mathrm{cns}})$ is combinatorial;
4. the graded set $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \mathrm{ht})$ is combinatorial.

4.2. Substitutions

Definition

Given a signature $\mathcal{S}$ and a set of variables V, an $\mathcal{S}, \mathsf{V}$-*substitution* is a function $\mathsf{V} \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$.

Let $\mathcal{S}$ be a signature, V be a set of variables, and σ be an $\mathcal{S}, \mathsf{V}$-substitution. The *domain* of σ is the set

$$\mathrm{Dom} \cdot \sigma := \{v \in \mathsf{V} : \sigma \cdot v \neq v\}.$$

Any subset S of $\mathsf{V} \times \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $(v, \mathfrak{t}) \in S$ and $(v, \mathfrak{t}') \in S$ implies $\mathfrak{t} = \mathfrak{t}'$ *specifies* the $\mathcal{S}, \mathsf{V}$-substitution $[S]$ defined, for any $v \in \mathsf{V}$, by

$$[S] \cdot v := \begin{cases} \mathfrak{t} & \text{if } (v, \mathfrak{t}) \in S, \\ v & \text{otherwise.} \end{cases}$$

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution

$$\sigma := [(\mathsf{v}_2, \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1), (\mathsf{v}_3, \mathsf{v}_3), (\mathsf{v}_5, \mathsf{v}_6)].$$

We have $\mathrm{Dom} \cdot \sigma = \{\mathsf{v}_2, \mathsf{v}_5\}$, $\sigma \cdot \mathsf{v}_2 = \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1$, $\sigma \cdot \mathsf{v}_5 = \mathsf{v}_6$, and $\sigma \cdot \mathsf{v}_i = \mathsf{v}_i$ for all $i \in \mathbb{N} \setminus \{2, 5\}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Given an $\mathcal{S}, \mathsf{V}$-substitution σ, the *extension* of σ is the function $\overline{\sigma} : \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \to \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ defined recursively, for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, by

$$\overline{\sigma} \cdot \mathfrak{t} := \begin{cases} \sigma \cdot v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ c \lfloor \overline{\sigma} \cdot \mathfrak{t}_1 \rfloor \dots \lfloor \overline{\sigma} \cdot \mathfrak{t}_n \rfloor & \text{otherwise, where } \mathfrak{t} = c\, \mathfrak{t}_1 \dots \mathfrak{t}_n. \end{cases}$$

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathsf{v}_1, \mathsf{c}_2\, \mathsf{v}_1\, \mathsf{v}_2), (\mathsf{v}_2, \mathsf{c}_1\, \mathsf{v}_5), (\mathsf{v}_3, \mathsf{v}_4)]$. We have

$$\overline{\sigma} \cdot \mathsf{c}_3\, \mathsf{v}_2 \lfloor \mathsf{c}_2\, \mathsf{v}_3\, \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_1\, \mathsf{c}_0 \rfloor = \mathsf{c}_3 \lfloor \mathsf{c}_1\, \mathsf{v}_5 \rfloor \lfloor \mathsf{c}_2\, \mathsf{v}_4 \lfloor \mathsf{c}_1\, \mathsf{v}_5 \rfloor \rfloor \lfloor \mathsf{c}_1\, \mathsf{c}_0 \rfloor.$$

An $\mathcal{S}, \mathsf{V}$-substitution σ is a *renaming* if σ is injective and for any $v \in \mathsf{V}$, $\sigma \cdot v = v'$ for a $v' \in \mathsf{V}$.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma := [(\mathsf{v}_1, \mathsf{v}_1), (\mathsf{v}_2, \mathsf{v}_3), (\mathsf{v}_3, \mathsf{v}_4), (\mathsf{v}_4, \mathsf{v}_2)]$. This $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution is a renaming and we have

$$\overline{\sigma} \cdot \mathsf{c}_3\, \mathsf{v}_2 \lfloor \mathsf{c}_2\, \mathsf{v}_3\, \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_1\, \mathsf{c}_0 \rfloor = \mathsf{c}_3\, \mathsf{v}_3 \lfloor \mathsf{c}_2\, \mathsf{v}_4\, \mathsf{v}_3 \rfloor \lfloor \mathsf{c}_1\, \mathsf{c}_0 \rfloor.$$

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *disjoint union* of two $\mathcal{S}, \mathsf{V}$-substitutions σ_1 and σ_2 such that $\mathrm{Dom} \cdot \sigma_1 \cap \mathrm{Dom} \cdot \sigma_2 = \emptyset$ is the $\mathcal{S}, \mathsf{V}$-substitution $\sigma_1 \sqcup \sigma_2$ defined, for any $v \in \mathsf{V}$, by

$$\lfloor \sigma_1 \sqcup \sigma_2 \rfloor \cdot v := \begin{cases} \sigma_1 \cdot v & \text{if } v \in \mathrm{Dom} \cdot \sigma_1, \\ \sigma_2 \cdot v & \text{otherwise.} \end{cases}$$

The *composition* of two $\mathcal{S}, \mathsf{V}$-substitutions σ_1 and σ_2 is the $\mathcal{S}, \mathsf{V}$-substitution $\sigma_1 \circ \sigma_2$ defined, for any $v \in \mathsf{V}$, by

$$\lfloor \sigma_1 \circ \sigma_2 \rfloor \cdot v := \overline{\sigma_1} \cdot \lfloor \sigma_2 \cdot v \rfloor.$$

Example

We have the following composition of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions:

$$[(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2)] \circ [(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_1), (\mathsf{v}_2, \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3)] = [(\mathsf{v}_1, \mathsf{c}_1 \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_3 \mathsf{c}_0 \mathsf{v}_2 \lfloor \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2 \rfloor), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2)].$$

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term, $v_1 \dots v_n$ be a sequence of pairwise distinct variables of V, $n \in \mathbb{N}$, and $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ be a sequence of $\mathcal{S}, \mathsf{V}$-terms. The *composition of* $\mathfrak{t}$ *and* $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ *on* $v_1 \dots v_n$ is the $\mathcal{S}, \mathsf{V}$-term

$$\mathfrak{t}[(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)] := \overline{[(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)]} \cdot \mathfrak{t}.$$

The $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}[(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)]$ is built by **replacing simultaneously** each occurrence of a variable $v_i \in \mathsf{V}$ of $\mathfrak{t}$ by $\mathfrak{t}_i$ for all $i \in [n]$.

Examples

We have the following compositions of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms:

- $\mathsf{c}_3\ \mathsf{v}_2\ \lfloor \mathsf{c}_2\ \mathsf{v}_1\ \mathsf{v}_2 \rfloor\ \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor [(\mathsf{v}_1, \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_1\ \mathsf{v}_5), (\mathsf{v}_4, \mathsf{v}_1)] = \mathsf{c}_3\ \lfloor \mathsf{c}_1\ \mathsf{v}_5 \rfloor\ \lfloor \mathsf{c}_2\ \mathsf{c}_0\ \lfloor \mathsf{c}_1\ \mathsf{v}_5 \rfloor \rfloor\ \lfloor \mathsf{c}_1 \mathsf{v}_3 \rfloor$;
- $\mathsf{c}_2\ \mathsf{v}_1\ \mathsf{v}_3 [(\mathsf{v}_1, \mathsf{v}_2), (\mathsf{v}_2, \mathsf{v}_2), (\mathsf{v}_3, \mathsf{v}_2), (\mathsf{v}_4, \mathsf{v}_2), (\mathsf{v}_5, \mathsf{v}_2)] = \mathsf{c}_2\ \mathsf{v}_2\ \mathsf{v}_2$.

Let $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ and $v \in \mathsf{V}$. The *partial composition of* $\mathfrak{t}$ *and* $\mathfrak{t}'$ *on* v is the $\mathcal{S}, \mathsf{V}$-term

$$\mathfrak{t} \curvearrowleft_v \mathfrak{t}' := \mathfrak{t}[(v, \mathfrak{t}')].$$

Let $\mathcal{S}$ be a signature.

A *labeled $\mathcal{S}$-substitution* is an $\mathcal{S}, \mathbb{N} \setminus \{0\}$-substitution.

The *labeled $\mathcal{S}$-substitution* of a sequence $\mathfrak{t}_1 \dots \mathfrak{t}_n$ of labeled $\mathcal{S}$-terms is the labeled $\mathcal{S}$-substitution

$$[\mathfrak{t}_1, \dots, \mathfrak{t}_n] := [(1, \mathfrak{t}_1), \dots, (n, \mathfrak{t}_n)].$$

Example

Let the labeled $\mathcal{S}_{\mathbb{N}^2}$-substitution σ defined by

$$\sigma := [2, 2, \mathsf{c}_2\, 3\, \mathsf{c}_0, \mathsf{c}_0, 5].$$

We have $\sigma \cdot 1 = 2$, $\sigma \cdot 2 = 2$, $\sigma \cdot 3 = \mathsf{c}_2\, 3\, \mathsf{c}_0$, $\sigma \cdot 4 = \mathsf{c}_0$, and for any $i \geqslant 5$, $\sigma \cdot i = i$.
Therefore, $\mathrm{Dom} \cdot \sigma = \{1, 3, 4\}$.

In this context of labeled $\mathcal{S}$-substitutions, for any labeled $\mathcal{S}$-term $\mathfrak{t}$ and any sequence $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$, $n \in \mathbb{N}$, of labeled $\mathcal{S}$-terms, let

$$\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n] := \mathfrak{t}[(1, \mathfrak{t}'_1), \dots, (n, \mathfrak{t}'_n)].$$

The composition of labeled $\mathcal{S}$-substitutions, when $\mathcal{S}$ is a signature, can be described in the following way.

Proposition [Composition of labeled substitutions]

For any signature $\mathcal{S}$, any sequences $\mathfrak{t}_1 \dots \mathfrak{t}_n$, $n \in \mathbb{N}$, and $\mathfrak{s}_1 \dots \mathfrak{s}_m$, $m \in \mathbb{N}$, of labeled $\mathcal{S}$-terms, and any labeled $\mathcal{S}$-term $\mathfrak{r}$ such that $m \geqslant \mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{r}$,

$$\overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n] \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_m]} \cdot \mathfrak{r} = \overline{\left[\overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n]} \cdot \mathfrak{s}_1, \dots, \overline{[\mathfrak{t}_1, \dots, \mathfrak{t}_n]} \cdot \mathfrak{s}_m\right]} \cdot \mathfrak{r}.$$

Exercise ●●○○○

Prove that the condition about the variable rank of $\mathfrak{r}$ is necessary in the previous proposition.

Exercise ●●●○○

Prove the previous proposition.

Theorem [Relations of compositions of labeled terms]

For any signature $\mathcal{S}$, the following relations hold:

- for any $n \in \mathbb{N}$, $i \in [n]$, and any sequence $\mathfrak{t}_1 \dots \mathfrak{t}_n$ of labeled $\mathcal{S}$-terms,

$$i\,[\mathfrak{t}_1, \dots, \mathfrak{t}_n] = \mathfrak{t}_i;$$

- for any $n \in \mathbb{N}$ and any labeled $\mathcal{S}$-term $\mathfrak{t}$ such that $n \geqslant \mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}$,

$$\mathfrak{t}[1, \dots, n] = \mathfrak{t};$$

- for any $n, m \in \mathbb{N}$, any labeled $\mathcal{S}$-term $\mathfrak{t}$, and any sequences $\mathfrak{t}'_1 \dots \mathfrak{t}'_n$ and $\mathfrak{t}''_1 \dots \mathfrak{t}''_m$ of labeled $\mathcal{S}$-terms such that $n \geqslant \mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}$, and $m \geqslant \mathrm{rk}_{\mathrm{v}} \cdot \mathfrak{t}'_i$ for all $i \in [n]$,

$$\underline{\mathfrak{t}[\mathfrak{t}'_1, \dots, \mathfrak{t}'_n]}\,[\mathfrak{t}''_1, \dots, \mathfrak{t}''_m] = \mathfrak{t}\left[\mathfrak{t}'_1\,[\mathfrak{t}''_1, \dots, \mathfrak{t}''_m], \dots, \mathfrak{t}'_n\,[\mathfrak{t}''_1, \dots, \mathfrak{t}''_m]\right].$$

Proof. The first two relations are immediate. The third one is a consequence of Proposition [Composition of labeled substitutions].

Exercise ○○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Exhibit the neutral element w.r.t. the composition of $\mathcal{S}, \mathsf{V}$-substitutions.

Exercise ●○○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Show that the composition of $\mathcal{S}, \mathsf{V}$-substitutions is not commutative.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Show that the composition of $\mathcal{S}, \mathsf{V}$-substitutions is associative.

Exercise ●●○○○

Let $\mathcal{S}$ be a signature, V be a set of variables, and let $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms. Describe a necessary and sufficient condition on both $\mathfrak{t}$ and $\mathfrak{t}'$ for the existence of two $\mathcal{S}, \mathsf{V}$-substitutions σ and σ' satisfying $\overline{\sigma} \cdot \mathfrak{t} = \mathfrak{t}'$ and $\overline{\sigma'} \cdot \mathfrak{t}' = \mathfrak{t}$.

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5. Term series

5.1. Formal series

Let $\mathbb{K}$ be a field.

For any set X, let $\mathbb{K}\langle X\rangle$ be the **$\mathbb{K}$-linear span** of X. By definition, X is a basis of $\mathbb{K}\langle X\rangle$.

Each element of $\mathbb{K}\langle X\rangle$ is a $\mathbb{K}, X$*-polynomial*.

Example

By setting $A := \{a, b\}$, the $\mathbb{K}$-linear combination

$$ab - ba + 2baa$$

is a $\mathbb{K}, A^*$-polynomial.

A $\mathbb{K}, X$-polynomial f is a **$\mathbb{K}$-linear combination** of elements of X so that f can be written as a **finite sum**

$$f = \sum_{x \in X} \lambda_x \, x,$$

where each $\lambda_x \in \mathbb{K}$, $x \in X$, is the *coefficient* of x in f.

The number of $x \in X$ such that $\lambda_x \neq 0$ is **finite**.

Definition

For any field $\mathbb{K}$ and set X, the *space of* $\mathbb{K}, X$*-series* is the $\mathbb{K}$-vector space $\mathbb{K}\langle\langle X\rangle\rangle$ defined as the dual space of $\mathbb{K}\langle X\rangle$.

Let $\mathbb{K}$ be a field and X be a set.

A $\mathbb{K}, X$-series is a **linear form** $\mathbf{f} : \mathbb{K}\langle X\rangle \to \mathbb{K}$ whose value on $x \in X$ is the *coefficient* $\mathbf{f} \cdot x$ of x in $\mathbf{f}$.

The *support* of $\mathbf{f} \in \mathbb{K}\langle\langle X\rangle\rangle$ is the set

$$\mathrm{Supp} \cdot \mathbf{f} := \{x \in X : \mathbf{f} \cdot x \neq 0\}.$$

The *canonical pairing* of a $\mathbb{K}, X$-polynomial f and a $\mathbb{K}, X$-series $\mathbf{f}$ is the element of $\mathbb{K}$ defined by

$$\langle f, \mathbf{f}\rangle := \mathbf{f} \cdot f = \sum_{x \in X} \lfloor f \cdot x \rfloor \lfloor \mathbf{f} \cdot x \rfloor.$$

Given $X' \subseteq X$, the *characteristic series of* X' is the $\mathbb{K}, X$-series $[X']$ defined, for any $x \in X$, by

$$[X'] \cdot x := [x \in X'].$$

For any $x \in X$, when there is no confusion, we denote by x the $\mathbb{K}, X$-series $[\{x\}]$.

Note in particular that, for any $x \in X$ and $\mathbf{f} \in \mathbb{K}\langle\langle X\rangle\rangle$, $\langle x, \mathbf{f}\rangle$ is the coefficient $\mathbf{f} \cdot x$ of x in $\mathbf{f}$.

Let $\mathbb{K}$ be a field and X be a set.

The *infinite sum notation* of $\mathbf{f} \in \mathbb{K}\langle\langle X \rangle\rangle$ allows us to formally write

$$\mathbf{f} = \sum_{x \in X} \langle x, \mathbf{f} \rangle \, x.$$

Examples

Let the alphabet $A := \{a, b\}$. Here are some $\mathbb{K}, A^*$-series:

- ☐ 0;
- ☐ aba;
- ☐ $ab + 2baa$;
- ☐ $1 + a + aa + aaa + \cdots$;
- ☐ $[A^*]$;
- ☐ $\sum_{w \in A^*} \lfloor \ell{\cdot}w \rfloor \, w$;
- ☐ $\sum_{w \in A^*} \lfloor \ell_a{\cdot}w \rfloor \, w$;
- ☐ $\sum_{w_1, w_2 \in A^*} w_1 \centerdot w_2$.

Exercise ○○○○○

Express the coefficients $\langle w, \mathbf{f} \rangle$ for all $w \in A^*$ such that $\ell{\cdot}w \leqslant 3$ for all $\mathbb{K}, A^*$-series $\mathbf{f}$ among the examples above.

Let $\mathbb{K}$ be a field.

For any alphabet A,

- $\mathbb{K}\langle\langle A^*\rangle\rangle$ is the $\mathbb{K}$-vector space of *noncommutative formal power series on* A;

  Examples

  The previous examples show noncommutative formal power series on A with $A = \{a, b\}$.

- $\mathbb{K}\langle\langle \mathrm{Mon}\cdot A\rangle\rangle$ is the $\mathbb{K}$-vector space of *formal power series on* A, where $\mathrm{Mon}\cdot A$ is the set of *monomials over* A that are **commutative words** on A.

  Examples

  Let $A = \{a, b\}$. In $\mathbb{K}\langle\langle \mathrm{Mon}\cdot A\rangle\rangle$, we have

  $$\sum_{w \in A^*} w = \epsilon + a + b + aa + ab + ba + bb + aaa + aab + aba + abb + baa + bab + bba + bbb + \cdots$$

  $$= \epsilon + a + b + aa + 2ab + bb + aaa + 3aab + 3abb + bbb + \cdots .$$

The linear function $\phi : \mathbb{K}\langle\langle A^*\rangle\rangle \to \mathbb{K}\langle\langle \mathrm{Mon}\cdot A\rangle\rangle$ satisfying, for any $w \in A^*$, $\phi \cdot w = \overline{w}$ where $\overline{w}$ is the **commutative image** of w, extends uniquely to a linear function between $\mathbb{K}\langle\langle A^*\rangle\rangle$ and $\mathbb{K}\langle\langle \mathrm{Mon}\cdot A\rangle\rangle$.

Let $\mathbb{K}$ be a field, X be a set, and $\mathrm{rk}_1 \dots \mathrm{rk}_k$, $k \geqslant 1$, be a sequence of functions such that for any $i \in [k]$, (X, rk_i) is a **graded set**.

Let the **alphabet of variables** $\mathsf{Z} := \{\mathsf{z}_i : i \in \mathbb{N} \setminus \{0\}\}$. To lighten the notation, we sometimes denote by z the variable z_1.

For any $\mathbf{f} \in \mathbb{K}\langle\langle X \rangle\rangle$, the $\mathrm{rk}_1 \dots \mathrm{rk}_k$-*trace* of $\mathbf{f}$ is the $\mathbb{K}, \mathrm{Mon}{\cdot}\mathsf{Z}$-series

$$\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k} \cdot \mathbf{f} := \sum_{x \in X} \langle x, \mathbf{f} \rangle \, \mathsf{z}_1^{\mathrm{rk}_1 \cdot x} \dots \mathsf{z}_k^{\mathrm{rk}_k \cdot x}.$$

Observe that the function $\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k}$ is **linear** and that $\mathrm{tr}_{\mathrm{rk}_1 \dots \mathrm{rk}_k} \cdot \mathbf{f}$ **may not be well-defined**.

Example

Let the $\mathbb{K}, \mathfrak{S}$-series

$$\mathbf{f} := \sum_{\sigma \in \mathfrak{S}} (-1)^{\mathrm{inv} \cdot \sigma} \, \sigma = \epsilon + 1 + 12 - 21 + 123 - 132 - 213 + 231 + 312 - 321 + \cdots .$$

By defining $\mathrm{des} \cdot \sigma$ as the number of descents of $\sigma \in \mathfrak{S}$, we have

$$\mathrm{tr}_{\ell \, \mathrm{des}} \cdot \mathbf{f} = \mathsf{z}_1^0 \mathsf{z}_2^0 + \mathsf{z}_1^1 \mathsf{z}_2^0 + \mathsf{z}_1^2 \mathsf{z}_2^0 + (-1) \mathsf{z}_1^2 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^0 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^1 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^1 + \mathsf{z}_1^3 \mathsf{z}_2^1 + (-1) \mathsf{z}_1^3 \mathsf{z}_2^2 + \cdots .$$

$$= 1 + \mathsf{z}_1 + \mathsf{z}_1^2 - \mathsf{z}_1^2 \mathsf{z}_2 + \mathsf{z}_1^3 - \mathsf{z}_1^3 \mathsf{z}_2^2 + \cdots .$$

Let $\mathbb{K}$ be a field and $\mathcal{G} := (X, \mathrm{rk})$ be a combinatorial graded set.

The *generating series of* $\mathcal{G}$ is the $\mathbb{K}, \mathrm{Mon}{\cdot}\{\mathrm{z}\}$-series

$$\langle \mathcal{G} \rangle := \mathrm{tr}_{\mathrm{rk}} \cdot [X].$$

Example

Consider the graded set $\mathcal{G} := (\{a, b\}^*, \ell)$. Since

$$[\{a, b\}^*] = \epsilon + a + b + aa + ab + ba + bb + aaa + aab + aba + abb + baa + bab + bba + bbb + \cdots,$$

we have

$$\langle \mathcal{G} \rangle = 1 + 2\mathrm{z} + 4\mathrm{z}^2 + 8\mathrm{z}^3 + \cdots.$$

Note that when $\mathcal{G}$ is not combinatorial, $\langle \mathcal{G} \rangle$ is not well-defined.

Exercise ○○○○○

Provide an example of a graded set (X, rk) such that $\langle (X, \mathrm{rk}) \rangle$ is not well-defined. Introduce another rank function rk' such that $\langle (X, \mathrm{rk}') \rangle$ is well-defined.