/ Term series

5.2. Products on series

For any set Y and $n \in \mathbb{N}$, an *n-operation on Y* is a function of type

$$\underbrace{Y \to \cdots \to Y}_{n \text{ times}} \to Y.$$

Note that a 0-operation is a **constant**.

Given a 2-operation θ on a set Y, for any $k \geqslant 1$, let $\theta^{(k)}$ be the k-operation defined recursively, for any $y_1, \dots, y_k \in Y$, by

$$\theta^{(k)} \cdot y_1 \cdot \cdots \cdot y_k := \begin{cases} y_1 & \text{if } k = 1, \\ \theta \cdot \underbracket{\theta^{(k-1)} \cdot y_1 \cdot \cdots \cdot y_{k-1}} \cdot y_k & \text{otherwise.} \end{cases}$$

Example

Let the alphabet $A := \{a, b\}$ and let the 2-operation θ on A^* satisfying $\theta \cdot w_1 \cdot w_2 = w_1 \centerdot w_2$ for any $w_1, w_2 \in A^*$.

This 2-operation θ has type $A^* \to A^* \to A^*$.

The 3-operation $\theta^{(3)}$ has type $A^* \to A^* \to A^* \to A^*$ and satisfies, for any $w_1, w_2, w_3 \in A^*$,

$$\theta^{(3)} \cdot w_1 \cdot w_2 \cdot w_3 = \theta \cdot \underbracket{\theta \cdot w_1 \cdot w_2} \cdot w_3 = (w_1 \centerdot w_2) \centerdot w_3 = w_1 \centerdot w_2 \centerdot w_3.$$

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X.

The *extension of θ on $\mathbb{K}\langle\langle X\rangle\rangle$* is the n-operation $\bar{\theta}$ on $\mathbb{K}\langle\langle X\rangle\rangle$ defined, for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$, by

$$\bar{\theta} \cdot \mathbf{f}_1 \cdot \dots \cdot \mathbf{f}_n := \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \dots \cdot x_n.$$

In other words, $\bar{\theta}$ is the multilinear function induced by θ.

Note that $\bar{\theta} \cdot \mathbf{f}_1 \cdot \dots \cdot \mathbf{f}_n$ may not be well-defined.

Example

Consider the alphabet A and the 2-operation θ on A^* of the previous example.

We have

$$\bar{\theta} \cdot \underbracket{a + bb} \cdot \underbracket{aa + 2b + ba} = aaa + 2ab + aba + bbaa + 2bbb + bbba,$$

and

$$\bar{\theta} \cdot \underbracket{\sum_{w_1 \in A^*} w_1} \cdot \underbracket{\sum_{w_2 \in A^*} w_2} = \sum_{w_1, w_2 \in A^*} \theta \cdot w_1 \cdot w_2 = \sum_{w_1, w_2 \in A^*} w_1 \mathbin{.} w_2 = \epsilon + 2a + 2b + 3aa + 3ab + 3ba + 3bb + \cdots.$$

Exercise ●○○○○

Let $\mathbb{K}$ be a field and X be a set. For any $n \in \mathbb{N}$ and $i \in \mathbb{N}$, let π_i be the n-operation on X such that for any $x_1, \dots, x_n \in X$, $\pi_i \cdot x_1 \cdot \cdots \cdot x_n := x_i$. Provide a description of $\overline{\pi_n} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n$ for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X \rangle\rangle$.

Exercise ●○○○○

Give an explicit example of a 2-operation θ on a set X such that $\bar{\theta} \cdot \mathbf{f}_1 \cdot \mathbf{f}_2$ is not well-defined for some $\mathbf{f}_1, \mathbf{f}_2 \in \mathbb{K}\langle\langle X \rangle\rangle$ where $\mathbb{K}$ is a field.

Exercise ●●○○○

Let the alphabet $A := \{a, b\}$. Let the 2-operation θ satisfying $\theta \cdot w_1 \cdot w_2 = w_1 \mathbin{\boldsymbol{.}} w_2$ for any $w_1, w_2 \in A^*$. Give, for any $w \in A^*$, an explicit expression for the coefficient

$$\left\langle w, \bar{\theta} \cdot \left\lfloor \sum_{w_1 \in A^*} \left\lfloor \ell_a \cdot w_1 \right\rfloor w_1 \right\rfloor \cdot \left\lfloor \sum_{w_2 \in A^*} \left\lfloor \ell_b \cdot w_2 \right\rfloor w_2 \right\rfloor \right\rangle.$$

Proposition [Multilinearity of extensions of operations]

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X. If the extension $\bar{\theta}$ of θ on $\mathbb{K}\langle\langle X\rangle\rangle$ is well-defined, then $\bar{\theta}$ is multilinear.

Proof. Let $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$, $\alpha \in \mathbb{K}$, $i \in [n]$, and $\mathbf{f}'_i \in \mathbb{K}\langle\langle X\rangle\rangle$. By multilinearity (in particular in its second argument) of the canonical pairing, we have

$$\bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \underbrace{\alpha \mathbf{f}_i + \mathbf{f}'_i} \cdot \ \dots \ \cdot \mathbf{f}_n = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \langle x_i, \alpha \mathbf{f}_i + \mathbf{f}'_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \alpha \langle x_i, \mathbf{f}_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n + \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{j \in [n] \setminus \{i\}} \langle x_j, \mathbf{f}_j \rangle \right) \langle x_i, \mathbf{f}'_i \rangle \, \theta \cdot x_1 \cdot \ \cdots \ \cdot x_n$$

$$= \alpha \, \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \mathbf{f}_i \cdot \ \dots \ \cdot \mathbf{f}_n + \bar{\theta} \cdot \mathbf{f}_1 \cdot \ \dots \ \cdot \mathbf{f}'_i \cdot \ \dots \ \cdot \mathbf{f}_n.$$

This shows the linearity of $\bar{\theta}$ in its i-th argument.

Proposition [Coefficients in series with extensions of operations]

Let $\mathbb{K}$ be a field, X be a set, and θ be an n-operation on X. If the extension $\overline{\theta}$ of θ on $\mathbb{K}\langle\langle X\rangle\rangle$ is well-defined, then for any $\mathbf{f}_1, \dots, \mathbf{f}_n \in \mathbb{K}\langle\langle X\rangle\rangle$ and $x \in X$,

$$\left\langle x, \overline{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \right\rangle = \sum_{(x_1, \dots, x_n) \in X^n} [x = \theta \cdot x_1 \cdot \cdots \cdot x_n] \prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle.$$

Proof. By multilinearity (in particular in its second argument) of the canonical pairing, we have

$$\left\langle x, \overline{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \right\rangle = \left\langle x, \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \cdots \cdot x_n \right\rangle$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \langle x, \theta \cdot x_1 \cdot \cdots \cdot x_n \rangle$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) [x = \theta \cdot x_1 \cdot \cdots \cdot x_n].$$

This shows the statement.

Let $\mathcal{G} := (X, \mathrm{rk})$ be a graded set. An n-operation θ is rk-*graded* if, for any $x_1, \dots, x_n \in X$,

$$\mathrm{rk} \cdot \lfloor \theta \cdot x_1 \cdot \cdots \cdot x_n \rfloor = \mathrm{rk} \cdot x_1 + \cdots + \mathrm{rk} \cdot x_n .$$

Examples

Let A be an alphabet.

☐ The concatenation operation $\centerdot$ on A^* is an ℓ-graded 2-operation. Indeed, for any $w_1, w_2 \in A^*$, $\ell \cdot \lfloor w_1 \centerdot w_2 \rfloor = \ell \cdot w_1 + \ell \cdot w_2$.

☐ The reversal function $w \mapsto w^{\mathrm{r}}$ on A^* where, for any $w \in A^*$ and $i \in [\ell \cdot w]$, $w^{\mathrm{r}} \cdot i := w \cdot \lfloor \ell \cdot w - i + 1 \rfloor$ is an ℓ-graded 1-operation.

☐ The 1-operation $\theta := w \mapsto w_{|A'}$ on A^* where, for any $w \in A^*$ and $A' \subseteq A$, $w_{|A'}$ is the subword of w consisting of its letters belonging to A', is not ℓ-graded. For instance, for $A := \{a, b, c\}$, $A' := \{a, b\}$, and $w := abca$, we have $\ell \cdot \lfloor \theta \cdot w \rfloor = \ell \cdot aba = 3 \neq 4 = \ell \cdot abca = \ell \cdot w$.

Nevertheless, θ is ℓ_x-graded for any $x \in A'$.

If θ is an rk-graded 2-operation, then for any $k \geqslant 1$, then $\theta^{(k)}$ is also rk-graded (follows by induction on k).

Proposition [Graded operations and traces]

Let $\mathbb{K}$ be a field, (X, rk) be a combinatorial graded set, and θ be an rk-graded n-operation on X. If the extension $\bar{\theta}$ of θ on $\mathbb{K}\langle\langle X \rangle\rangle$ is well-defined, then

$$\mathrm{tr}_{\mathrm{rk}} \cdot \underline{\lfloor \bar{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \rfloor} = \prod_{i \in [n]} \mathrm{tr}_{\mathrm{rk}} \cdot \mathbf{f}_i$$

where $\prod$ is the iterated product on monomials.

Proof. By definition of $\bar{\theta}$ and $\mathrm{tr}_{\mathrm{rk}}$, by linearity of $\mathrm{tr}_{\mathrm{rk}}$, and since θ is rk-graded, we have

$$\mathrm{tr}_{\mathrm{rk}} \cdot \underline{\lfloor \bar{\theta} \cdot \mathbf{f}_1 \cdot \cdots \cdot \mathbf{f}_n \rfloor} = \mathrm{tr}_{\mathrm{rk}} \cdot \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \theta \cdot x_1 \cdot \cdots \cdot x_n$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathrm{tr}_{\mathrm{rk}} \cdot \underline{\lfloor \theta \cdot x_1 \cdot \cdots \cdot x_n \rfloor} = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathsf{z}^{\mathrm{rk} \cdot \underline{\lfloor \theta \cdot x_1 \cdot \cdots \cdot x_n \rfloor}}$$

$$= \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \right) \mathsf{z}^{\mathrm{rk} \cdot x_1 + \cdots + \mathrm{rk} \cdot x_n} = \sum_{(x_1, \dots, x_n) \in X^n} \left(\prod_{i \in [n]} \langle x_i, \mathbf{f}_i \rangle \, \mathsf{z}^{\mathrm{rk} \cdot x_i} \right)$$

$$= \prod_{i \in [n]} \left(\sum_{x \in X} \langle x, \mathbf{f}_i \rangle \, \mathsf{z}^{\mathrm{rk} \cdot x} \right) = \prod_{i \in [n]} \mathrm{tr}_{\mathrm{rk}} \cdot \mathbf{f}_i .$$

/ Term series

5.3. Enumeration

The *integer sequence* of a combinatorial graded set $\mathcal{G} := (X, \mathrm{rk})$ is the sequence $\left(\# \cdot \lfloor \mathcal{G} \cdot n \rfloor\right)_{n \in \mathbb{N}}$.

An approach to compute the entries of the integer sequence of $\mathcal{G}$ consists in providing a description of the generating series $\langle \mathcal{G} \rangle$ of $\mathcal{G}$.

Here are the steps:

1. Set $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$.
2. Consider a collection $\{\theta_i : i \in [k]\}$ of rk-graded n_i-operation on X, where $n_i \in \mathbb{N}$, $i \in [k]$, $k \in \mathbb{N}$.
3. Express the characteristic series $[X]$ of X via a system of equations using the extensions $\overline{\theta_i}$ on $\mathbb{K}\langle\langle X \rangle\rangle$ of the θ_i, $i \in [k]$.
4. By applying the rk-trace function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the equations of the system, transform the previous system of equations of series of $\mathbb{K}\langle\langle X \rangle\rangle$ into a system of equations on formal power series of $\mathbb{K}\langle\langle \mathrm{Mon} \cdot \mathsf{Z} \rangle\rangle$.
5. Use the previous equations to express, for any $n \in \mathbb{N}$, the coefficient $\langle \mathsf{z}^n, \langle \mathcal{G} \rangle \rangle$.

It follows from Proposition [Graded operations and traces] that for any $n \in \mathbb{N}$, $\langle \mathsf{z}^n, \langle \mathcal{G} \rangle \rangle = \# \cdot \lfloor \mathcal{G} \cdot n \rfloor$ as expected.

Let us give some examples fitting in this framework, involving the enumeration of some families of paths.

A *path* is a nonempty word on $\mathbb{N}$. The *rank* $\mathrm{rk} \cdot p$ of a path p is $\ell \cdot p - 1$.

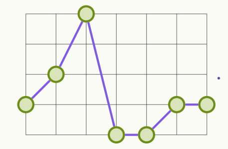
Let P be the graded set whose underlying set is the set of paths and rank function is rk.

A path p of rank $n-1$, $n \geqslant 1$, is drawn as the **set of points** $\{(i-1, p \cdot i) : i \in [n]\}$, where any pair of adjacent points is connected by a **step**.

Note that the **rank** of a path p is the **number of steps** of p.

Example

The path 1240011 has rank 6 and is depicted as



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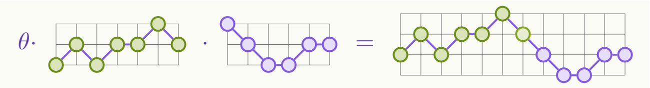
The *concatenation* of paths is the 2-operation θ defined for any $p_1 \in \mathsf{P}\cdot\lfloor n_1 - 1 \rfloor$, $n_1 \geqslant 1$, and $p_2 \in \mathsf{P}\cdot\lfloor n_2 - 1 \rfloor$, $n_2 \geqslant 1$, by

$$\theta \cdot p_1 \cdot p_2 := \begin{cases} \lfloor \uparrow_k \cdot p_1' \rfloor \centerdot p_2 & \text{if } p_1 \cdot n_1 \leqslant p_2 \cdot 1, \\ p_1 \centerdot \lfloor \uparrow_k \cdot p_2' \rfloor & \text{otherwise,} \end{cases}$$

where for any $w \in \mathbb{N}^*$ and $j \in \mathbb{N}$, $\uparrow_j \cdot w$ is the word obtained by incrementing by j each letter of w, $k := |p_1 \cdot n_1 - p_2 \cdot 1|$, and p_1' (resp. p_2') is the word obtained by deleting the last (resp. first) letter of p_1 (resp. p_2).

Example



Exercise ●○○○○

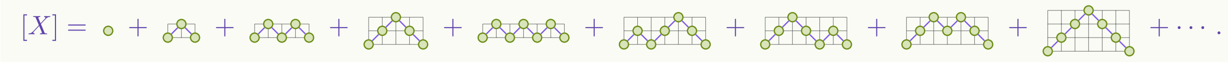
Prove that the 2-operation θ on paths is rk-graded.

Exercise ●●●○○

Prove that the 2-operation θ on paths is associative.

Let DP be the graded set of *Dyck paths*, defined as the sub-graded set of P containing the paths whose first and last letters are 0, and obtained by iterated concatenation via θ of the paths 01 and 10.

By denoting by X the underlying set of DP, we have

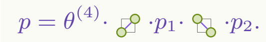


$$[X] = \circ + \cdots + \cdots .$$

Lemma [Decomposition of Dyck paths]

If p is a Dyck path, then

1. either $p = \circ$;
2. or there exists a unique pair (p_1, p_2) of Dyck paths such that



$$p = \theta^{(4)} \cdot \; \cdot p_1 \cdot \; \cdot p_2 .$$

Exercise ●●○○○

Prove the previous lemma.

From Lemma [Decomposition of Dyck paths], it follows that the characteristic series of DP satisfies the equation in $\mathbb{K}\langle\langle \mathsf{P} \rangle\rangle$

$[X] = \circ + \overline{\theta^{(4)}} \cdot$  $\cdot [X] \cdot$  $\cdot [X].$

By considering the images by the function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the previous equation, and since θ is rk-graded,

$$\langle \mathsf{DP} \rangle = 1 + \mathsf{z}^2 \langle \mathsf{DP} \rangle^2.$$

From this, we obtain that for any $n \in \mathbb{N}$,

$$\langle \mathsf{z}^n, \langle \mathsf{DP} \rangle \rangle = \begin{cases} 1 & \text{if } n = 0, \\ \sum\limits_{n_1, n_2 \in \mathbb{N}} [n_1 + n_2 = n - 2] \langle \mathsf{z}^{n_1}, \langle \mathsf{DP} \rangle \rangle \langle \mathsf{z}^{n_2}, \langle \mathsf{DP} \rangle \rangle & \text{otherwise.} \end{cases}$$

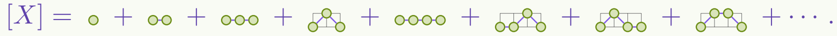
By computing the first values, we obtain that the integer sequence of DP starts by

$$1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, 429, 0.$$

By deleting the zeros, this is the Catalan integer sequence (A000108).

Let MP be the graded set of *Motzkin paths*, defined as the sub-graded set of P containing the paths whose first and last letters are 0, and obtained by iterated concatenation via θ of the paths 01, 10, and 00.

By denoting by X the underlying set of MP, we have

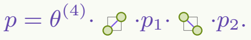


Lemma [Decomposition of Motzkin paths]

If p is a Motzkin path, then

1. either $p =$ ○ ;
2. or there exists a unique Motzkin path p_1 such that

3. or there exists a unique pair (p_1, p_2) of Motzkin paths such that



Exercise ●●○○○

Prove the previous lemma.

From Lemma [Decomposition of Motzkin paths], it follows that the characteristic series of MP satisfies the equation in $\mathbb{K}\langle\langle \mathsf{P}\rangle\rangle$

$[X] =$  $+ \bar{\theta} \cdot$  $\cdot [X] + \overline{\theta^{(4)}} \cdot$  $\cdot [X] \cdot$  $\cdot [X].$

By considering the images by the function $\mathrm{tr}_{\mathrm{rk}}$ on both sides of the previous equation, and since θ is rk-graded,

$$\langle \mathsf{MP} \rangle = 1 + \mathsf{z}\langle \mathsf{MP} \rangle + \mathsf{z}^2 \langle \mathsf{MP} \rangle^2.$$

From this, we obtain that for any $n \in \mathbb{N}$,

$$\langle \mathsf{z}^n, \langle \mathsf{MP} \rangle \rangle = \begin{cases} 1 & \text{if } n = 0, \\ \langle \mathsf{z}^{n-1}, \langle \mathsf{MP} \rangle \rangle + \sum\limits_{n_1, n_2 \in \mathbb{N}} [n_1 + n_2 = n - 2] \langle \mathsf{z}^{n_1}, \langle \mathsf{MP} \rangle \rangle \langle \mathsf{z}^{n_2}, \langle \mathsf{MP} \rangle \rangle & \text{otherwise.} \end{cases}$$

By computing the first values, we obtain that the integer sequence of MP starts by

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572.$$

This is the sequence of **Motzkin numbers** (A001006).

5.4. Term series and substitutions

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ be an $\mathcal{S}, \mathsf{V}$-term and $v_1 \dots v_n$, $n \in \mathbb{N}$, be a sequence of pairwise distinct variables of V.

The *$\mathfrak{t}, v_1 \dots v_n$-grafting operation* is the n-operation $\theta_{\mathfrak{t}, v_1 \dots v_n}$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ defined, for any $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$, by

$$\theta_{\mathfrak{t}, \mathsf{v}_1 \dots \mathsf{v}_n} \cdot \mathfrak{t}'_1 \cdot \dots \cdot \mathfrak{t}'_n := \mathfrak{t}[\{(v_1, \mathfrak{t}'_1), \dots, (v_n, \mathfrak{t}'_n)\}].$$

Example

By considering the signature $\mathcal{S}_{\mathbb{N}^2}$, the set of variables $\mathsf{V}_{\mathbb{N}}$, and by setting $\mathfrak{t} := \mathsf{c}_2 \lfloor \underline{\mathsf{c}_1 \mathsf{v}_3} \rfloor \lfloor \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2} \rfloor$, we have

$$\theta_{\mathfrak{t}, \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3} \cdot \lfloor \underline{\mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_2 \mathsf{v}_4} \rfloor \cdot \lfloor \underline{\mathsf{c}_1 \mathsf{v}_1} \rfloor \cdot \lfloor \underline{\mathsf{c}_2 \mathsf{v}_2 \lfloor \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_6} \rfloor} \rfloor = \mathsf{c}_2 \lfloor \underline{\mathsf{c}_1 \lfloor \underline{\mathsf{c}_2 \mathsf{v}_2 \lfloor \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_6} \rfloor} \rfloor} \rfloor \lfloor \underline{\mathsf{c}_2 \lfloor \underline{\mathsf{c}_3 \mathsf{v}_2 \mathsf{v}_2 \mathsf{v}_4} \rfloor \lfloor \underline{\mathsf{c}_1 \mathsf{v}_1} \rfloor} \rfloor.$$

If $\mathbb{K}$ is a field, an $\mathcal{S}, \mathsf{V}$-*term series* is a $\mathbb{K}, \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$-series.

Such $\mathfrak{t}$-grafting operations, together with $\mathcal{S}, \mathsf{V}$-term series, can be used to describe characteristic series of some families of $\mathcal{S}, \mathsf{V}$-terms in order to enumerate them w.r.t. some adequate rank functions.

Let $\mathcal{S} := (C, \mathrm{ar})$ be a signature and V be a set of variables.

Let t be an $\mathcal{S}, \mathsf{V}$-term, $v_1 \dots v_n$, $n \in \mathbb{N}$, be a sequence of pairwise distinct variables of V.

By setting $V := \{v_1, \dots, v_n\}$, for any $\mathcal{S}, \mathsf{V}$-terms $\mathsf{t}'_1, \dots, \mathsf{t}'_n$,

- ☐ for any $v \in \mathsf{V}$,

$$\ell_v \cdot \underbracket{\theta_{\mathsf{t}, v_1 \dots v_n} \cdot \mathsf{t}'_1 \cdot \; \cdots \; \cdot \mathsf{t}'_n} = [v \notin V] \, \underbracket{\ell_v \cdot \mathsf{t}} + \sum_{i \in [n]} \underbracket{\ell_{v_i} \cdot \mathsf{t}} \, \underbracket{\ell_v \cdot \mathsf{t}'_i}.$$

  When $\mathrm{Vars} \cdot \mathsf{t} = V$ and t is linear, the n-operation $\theta_{\mathsf{t}, v_1 \dots v_n}$ is **ℓ_v-graded**. In this case, $\theta_{\mathsf{t}, v_1 \dots v_n}$ is also **ℓ_{var}-graded**;

- ☐ for any $c \in C$,

$$\ell_c \cdot \underbracket{\theta_{\mathsf{t}, v_1 \dots v_n} \cdot \mathsf{t}'_1 \cdot \; \cdots \; \cdot \mathsf{t}'_n} = \ell_c \cdot \mathsf{t} + \sum_{i \in [n]} \underbracket{\ell_{v_i} \cdot \mathsf{t}} \, \underbracket{\ell_c \cdot \mathsf{t}'_i}.$$

  When $\mathrm{Vars} \cdot \mathsf{t} = V$, t is linear, and $\ell_c \cdot \mathsf{t} = 0$, the n-operation $\theta_{\mathsf{t}, v_1 \dots v_n}$ is **ℓ_c-graded**.

Exercise ●●●○○

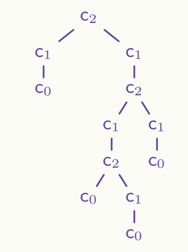
Let $\mathcal{S}$ be a signature and V be a set of variables. Provide a necessary and sufficient condition for a rank function rk, an $\mathcal{S}, \mathsf{V}$-term t, and a sequence $v_1 \dots v_n$, $n \in \mathbb{N}$, of pairwise distinct variables of V for the fact that the n-operation $\theta_{\mathsf{t}, v_1 \dots v_n}$ is a rk-graded.

An *alternating unary binary tree* is an $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-term $\mathfrak{t}$ such that

1. any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_1, \mathsf{c}_2\}$;
2. for any positions u and $u \cdot j$ in $\mathfrak{t}$ such that $j \in \mathbb{N} \setminus \{0\}$, the decorations of the internal nodes at positions u and $u \cdot j$ in $\mathfrak{t}$ are different.

Example

Here is an alternating unary binary tree:



Let AUB be the graded set whose underlying set is the set of alternating unary binary trees and rank function is ℓ_{c_0}. Recall that this function sends each alternating unary binary tree $\mathfrak{t}$ to the number of internal nodes of $\mathfrak{t}$ decorated by c_0.

Exercise ●●○○○

Prove that the graded set AUB is combinatorial.

The 1-*grafting* of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms is the 1-operation $\theta_1 := \theta_{\mathsf{c}_1 \mathsf{v}_1, \mathsf{v}_1}$.

Similarly, the 2-*grafting* of $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms is the 2-operation $\theta_2 := \theta_{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_2, \mathsf{v}_1 \mathsf{v}_2}$.

Examples

On alternating unary binary trees, we have

- ☐ $\theta_1 \cdot \underbracket{\mathsf{c}_2 \underbracket{\mathsf{c}_1 \mathsf{c}_0} \underbracket{\mathsf{c}_1 \mathsf{c}_0}} = \mathsf{c}_1 \underbracket{\mathsf{c}_2 \underbracket{\mathsf{c}_1 \mathsf{c}_0} \underbracket{\mathsf{c}_1 \mathsf{c}_0}}$;
- ☐ $\theta_2 \cdot \underbracket{\mathsf{c}_1 \mathsf{c}_0} \cdot \underbracket{\mathsf{c}_1 \underbracket{\mathsf{c}_2 \mathsf{c}_0 \mathsf{c}_0}} = \mathsf{c}_2 \underbracket{\mathsf{c}_1 \mathsf{c}_0} \underbracket{\mathsf{c}_1 \underbracket{\mathsf{c}_2 \mathsf{c}_0 \mathsf{c}_0}}$.

Exercise ●○○○○

Prove that the 1-operation θ_1 and the 2-operation θ_2 on $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms are ℓ_{c_0}-graded.

Exercise ●○○○○

Prove that the 2-operation θ_2 on $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}$-terms is not associative.

By denoting by X the underlying set of AUB, we have

$$[X] = \mathsf{c}_0 + \mathsf{c}_1\mathsf{c}_0 + \mathsf{c}_2\mathsf{c}_0\mathsf{c}_0 + \mathsf{c}_2\mathsf{c}_0\underbracket{\mathsf{c}_1\mathsf{c}_0} + \mathsf{c}_2\underbracket{\mathsf{c}_1\mathsf{c}_0}\mathsf{c}_0 + \mathsf{c}_2\underbracket{\mathsf{c}_1\mathsf{c}_0}\underbracket{\mathsf{c}_1\mathsf{c}_0} + \mathsf{c}_1\underbracket{\mathsf{c}_2\mathsf{c}_0\mathsf{c}_0} + \mathsf{c}_1\underbracket{\mathsf{c}_2\mathsf{c}_0\underbracket{\mathsf{c}_1\mathsf{c}_0}}$$

$$+\mathsf{c}_1\underbracket{\mathsf{c}_2\underbracket{\mathsf{c}_1\mathsf{c}_0}\mathsf{c}_0} + \mathsf{c}_1\underbracket{\mathsf{c}_2\underbracket{\mathsf{c}_1\mathsf{c}_0}\underbracket{\mathsf{c}_1\mathsf{c}_0}} + \mathsf{c}_2\mathsf{c}_0\underbracket{\mathsf{c}_1\underbracket{\mathsf{c}_2\mathsf{c}_0\mathsf{c}_0}} + \mathsf{c}_2\mathsf{c}_0\underbracket{\mathsf{c}_1\underbracket{\mathsf{c}_2\mathsf{c}_0\underbracket{\mathsf{c}_1\mathsf{c}_0}}} + \cdots.$$

Lemma [Decomposition of alternating unary binary trees]

If $\mathfrak{t}$ is an alternating unary binary tree, then

1. either $\mathfrak{t} = \mathsf{c}_0$;
2. or there exists a unique alternating unary binary tree $\mathfrak{t}_1$ having root decorated by c_0 or by c_2 such that $\mathfrak{t} = \theta_1 \cdot \mathfrak{t}_1$;
3. or there exists a unique pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of alternating unary binary trees having roots decorated by c_0 or by c_1 such that $\mathfrak{t} = \theta_2 \cdot \mathfrak{t}_1 \cdot \mathfrak{t}_2$.

Exercise ●○○○○

Prove the previous lemma.

From Lemma [Decomposition of alternating unary binary trees], it follows that the characteristic series of AUB satisfies the following system of equations in $\mathbb{K}\langle\langle \mathfrak{T}\cdot\mathcal{S}_{\mathbb{N}^2}\cdot\mathsf{V}_{\mathbb{N}}\rangle\rangle$:

$$[X] = \mathsf{c}_0 + \overline{\theta_1}\cdot[X_2] + \overline{\theta_2}\cdot[X_1]\cdot[X_1],$$

$$[X_1] = \mathsf{c}_0 + \overline{\theta_1}\cdot[X_2],$$

$$[X_2] = \mathsf{c}_0 + \overline{\theta_2}\cdot[X_1]\cdot[X_1],$$

where X_1 is the subset of X consisting of alternating unary binary trees whose roots are decorated by c_0 or by c_1, and where X_2 is the subset of X consisting of alternating unary binary trees whose roots are decorated by c_0 or by c_2.

By considering the images by the function $\mathrm{tr}_{\ell_{\mathsf{c}_0}}$ on both sides of the previous equations of the system, and since θ_1 and θ_2 are ℓ_{c_0}-graded,

$$\langle \mathsf{AUB}\rangle = \mathsf{z} + \mathbf{f}_2 + \mathbf{f}_1^2,$$

$$\mathbf{f}_1 = \mathsf{z} + \mathbf{f}_2,$$

$$\mathbf{f}_2 = \mathsf{z} + \mathbf{f}_1^2,$$

where $\mathbf{f}_1 := \mathrm{tr}_{\ell_{\mathsf{c}_0}}\cdot[X_1]$ and $\mathbf{f}_2 := \mathrm{tr}_{\ell_{\mathsf{c}_0}}\cdot[X_2]$.

From the previous system of equations, we obtain

$$\langle \mathsf{AUB} \rangle = 2\mathsf{f}_2 \qquad \text{and} \qquad \mathsf{f}_2 = \mathsf{z} + \mathsf{z}^2 + 2\mathsf{z}\mathsf{f}_2 + \mathsf{f}_2^2$$

so that, for any $n \in \mathbb{N}$,

$$\langle \mathsf{z}^n, \langle \mathsf{AUB} \rangle \rangle = 2 \langle \mathsf{z}^n, \mathsf{f}_2 \rangle \qquad \text{and} \qquad \langle \mathsf{z}^n, \mathsf{f}_2 \rangle = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 2 \langle \mathsf{z}^{n-1}, \mathsf{f}_2 \rangle + \displaystyle\sum_{n_1, n_2 \in [n-1]} [n_1 + n_2 = n] \, \langle \mathsf{z}^{n_1}, \mathsf{f}_2 \rangle \langle \mathsf{z}^{n_2}, \mathsf{f}_2 \rangle & \text{otherwise.} \end{cases}$$

By computing the first values, we obtain that the integer sequence of AUB starts by

$$0, 2, 8, 32, 160, 896, 5376, 33792, 219648, 1464320, 9957376, 68796416.$$

Exercise ●●●○○

Give a very simple combinatorial argument showing that for any $n \geqslant 2$, $\langle \mathsf{z}^n, \langle \mathsf{AUB} \rangle \rangle = 2^{n+1} \mathrm{cat} \cdot \lfloor n - 1 \rfloor$, where, for any $k \in \mathbb{N}$, $\mathrm{cat} \cdot k = \binom{2k}{k} \frac{1}{k+1}$ is the number of binary trees with k internal nodes.

Exercise ●●●○○

Let X be the set of $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-terms $\mathfrak{t}$ such that any internal node of $\mathfrak{t}$ is decorated on $\{\mathsf{c}_0, \mathsf{c}_2, \mathsf{c}_3\}$ and any internal decorated by c_3 has no child which is an internal node decorated by c_2.

1. Prove that the graded set (X, ℓ_{c_0}) is combinatorial.
2. Define some grafting operations allowing us to decompose any $\mathcal{S}_{\mathbb{N}^2}, \emptyset$-term of X in a recursive way.
3. Provide a system of equations for $[X]$ using the extensions of the previous grafting operations.
4. Deduce from this system of equations a system of equations for the generating series $\langle (X, \ell_{\mathsf{c}_0}) \rangle$ of (X, ℓ_{c_0}).
5. Deduce from this system of equations a recurrence formula to compute the terms of the integer sequence of (X, ℓ_{c_0}).