Exercise

Let N be the set of variables $\mathbb{N} \setminus \{0\}$ and, for any $m \in \mathbb{N}$, $\mathcal{S}_m$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_{2,i} : i \in [\![m]\!]\}$.

For any $m \in \mathbb{N}$, let the TRS $\mathsf{FCat}_m := (\mathcal{S}_m, N, \to_m)$ such that $\to_m$ satisfies

$$\mathsf{c}_{2,i+j} \lfloor \mathsf{c}_i 12 \rfloor 3 \to_m \mathsf{c}_{2,i} 1 \lfloor \mathsf{c}_{2,j} 23 \rfloor$$

for any $i, j \in [\![m]\!]$ with $i + j \in [\![m]\!]$.

Describe a system of equations in $\mathbb{K}\langle\langle \mathfrak{T} \cdot \mathcal{S}_m \cdot N \rangle\rangle$ for the planar normal forms of FCat_m, $m \in \mathbb{N}$.

Exercise

Let N be the set of variables $\mathbb{N} \setminus \{0\}$ and $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{\mathsf{c}_{2,0}, \mathsf{c}_{2,1}, \mathsf{c}_{2,2}\}$.

Let the TRS $\mathsf{Schr} := (\mathcal{S}, N, \to)$ such that $\to$ satisfies

$$\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,2} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,1} 23 \rfloor, \quad \mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,2} 23 \rfloor, \quad \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,0} 1 \lfloor \mathsf{c}_{2,1} 23 \rfloor,$$

$$\mathsf{c}_{2,0} \lfloor \mathsf{c}_{2,2} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,1} \lfloor \mathsf{c}_{2,1} 12 \rfloor 3 \to \mathsf{c}_{2,1} 1 \lfloor \mathsf{c}_{2,0} 23 \rfloor, \quad \mathsf{c}_{2,2} \lfloor \mathsf{c}_{2,0} 12 \rfloor 3 \to \mathsf{c}_{2,2} 1 \lfloor \mathsf{c}_{2,2} 23 \rfloor.$$

Describe a system of equations in $\mathbb{K}\langle\langle \mathfrak{T} \cdot \mathcal{S} \cdot N \rangle\rangle$ for the planar normal forms of Schr.

7.2. Reduction relations

Let $\mathcal{A} := (X, \Rightarrow)$ be an ARS.

A binary relation $\rightsquigarrow$ is a *termination witness* of $\mathcal{A}$ if the ARS $\mathcal{A}' := (X, \rightsquigarrow)$ is terminating and $\mathcal{A}$ is a sub-ARS of $\mathcal{A}'$.

Proposition [Termination witnesses]

An ARS $\mathcal{A}$ is terminating iff $\mathcal{A}$ admits a termination witness.

Proof. Let us denote by $\Rightarrow$ the rewrite relation of $\mathcal{A}$.

Assume that $\mathcal{A}$ is terminating. In this case, $\Rightarrow$ is trivially a termination witness of $\mathcal{A}$.

Conversely, assume that $\mathcal{A}$ admits a termination witness $\rightsquigarrow$. Assume that $(u_i)_{i \in \mathbb{N}}$ is an infinite rewrite sequence in $\mathcal{A}$. For any $i \in \mathbb{N}$, $u_i \Rightarrow u_{i+1}$. Since, by hypothesis, $\mathcal{A}$ is a sub-ARS of $\mathcal{A}' := (X, \rightsquigarrow)$, we have $\Rightarrow \subseteq \rightsquigarrow$ so that, for any $i \in \mathbb{N}$, $u_i \rightsquigarrow u_{i+1}$. This shows that $(u_i)_{i \in \mathbb{N}}$ is also an infinite rewrite sequence in $\mathcal{A}'$. This contradicts our hypothesis that $\mathcal{A}'$ is terminating, showing that such an infinite rewrite sequence $(u_i)_{i \in \mathbb{N}}$ in $\mathcal{A}$ does not exist. Therefore, $\mathcal{A}$ is terminating.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a binary relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

The binary relation $\rightsquigarrow$ is *compatible from factors* if $\rightsquigarrow$ satisfies the two following properties:

(CF1) for any $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $i \in [n]$, $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$,

$$\mathfrak{t}_i \rightsquigarrow \mathfrak{t}'_i \quad \text{implies} \quad c\,\mathfrak{t}_1 \dots \mathfrak{t}_i \dots \mathfrak{t}_n \rightsquigarrow c\,\mathfrak{t}_1 \dots \mathfrak{t}'_i \dots \mathfrak{t}_n;$$

(CF2) for any $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ and any $\mathcal{S}, \mathsf{V}$-substitution σ,

$$\mathfrak{t} \rightsquigarrow \mathfrak{t}' \quad \text{implies} \quad \overline{\sigma}{\cdot}\mathfrak{t} \rightsquigarrow \overline{\sigma}{\cdot}\mathfrak{t}'.$$

Proposition [Compatibility from factors]

Let $\mathcal{S}$ be a signature and V be a set of variables. A binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ is compatible from factors iff for any holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and any $\mathcal{S}, \mathsf{V}$-substitution σ,

$$\mathfrak{t} \rightsquigarrow \mathfrak{t}' \quad \text{implies} \quad \blacktriangle{\cdot}\mathfrak{s}{\cdot}\mathfrak{t}{\cdot}\sigma \rightsquigarrow \blacktriangle{\cdot}\mathfrak{s}{\cdot}\mathfrak{t}'{\cdot}\sigma.$$

Definition

Let $\mathcal{S}$ be a signature and V be a set of variables. A binary relation $\rightsquigarrow$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ is a *reduction relation* on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ if the ARS $(\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}, \rightsquigarrow)$ is terminating and $\rightsquigarrow$ is compatible from factors.

Let $\mathcal{S}$ be a signature and V be a set of variables.

The *length relation* is the binary relation $\rightsquigarrow_\ell$ on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathrm{V}$ defined, for any $\mathcal{S},\mathrm{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, by $\mathfrak{t} \rightsquigarrow_\ell \mathfrak{t}'$ if $\ell\cdot\mathfrak{t} > \ell\cdot\mathfrak{t}'$ and, for any $v \in \mathrm{V}$, $\ell_v\cdot\mathfrak{t} \geqslant \ell_v\cdot\mathfrak{t}'$.

Examples

On $\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}$-terms, we have

- $\mathfrak{t} := \mathsf{c}_2 \underline{\mathsf{c}_4 \mathsf{v}_2 \mathsf{v}_1 \mathsf{v}_1 \mathsf{v}_1} \mathsf{v}_3 \rightsquigarrow_\ell \mathsf{c}_3 \underline{\mathsf{c}_2 \mathsf{v}_1 \mathsf{v}_1} \mathsf{c}_0 \mathsf{c}_0 := \mathfrak{t}'$ because $\ell\cdot\mathfrak{t} = 7 > 6 = \ell\cdot\mathfrak{t}'$, $\ell_{\mathsf{v}_1}\cdot\mathfrak{t} = 3 \geqslant 2 = \ell_{\mathsf{v}_1}\cdot\mathfrak{t}'$, $\ell_{\mathsf{v}_2}\cdot\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}\cdot\mathfrak{t}'$, and $\ell_{\mathsf{v}_3}\cdot\mathfrak{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_3}\cdot\mathfrak{t}'$.
- By setting $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_2$ and $\mathfrak{t}' := \mathsf{c}_2 \mathsf{v}_2 \mathsf{v}_3$, we have $\mathfrak{t} \not\rightsquigarrow_\ell \mathfrak{t}'$ and $\mathfrak{t}' \not\rightsquigarrow_\ell \mathfrak{t}$.

Proposition [Length reduction relation]

For any signature $\mathcal{S}$ and any set of variables V, $\rightsquigarrow_\ell$ is a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathrm{V}$.

Exercise ●●●○○

Prove the previous proposition.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be a TRS.

A binary relation $\rightsquigarrow$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ is $\mathcal{T}$*-compatible* if $\to \subseteq \rightsquigarrow$.

Theorem [Compatible reduction relations and termination]

A TRS $\mathcal{T}$ is terminating iff there exists a $\mathcal{T}$-compatible reduction relation.

Proof. Let $\mathcal{S}$ be the underlying signature of $\mathcal{T}$, V the underlying set of variables of $\mathcal{T}$, and $\to$ be the elementary rewrite relation of $\mathcal{T}$.

Assume that $\mathcal{T}$ is terminating. By Propositions [Rewrite relation of a TRS] and [Compatibility from factors], $\Rightarrow$ is compatible from factors. Moreover, $\Rightarrow$ is trivially $\mathcal{T}$-compatible. Since by hypothesis, the ARS $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \Rightarrow)$ is terminating, $\Rightarrow$ is a $\mathcal{T}$-compatible reduction relation.

Assume that $\rightsquigarrow$ is a $\mathcal{T}$-compatible reduction relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$. Since $\rightsquigarrow$ is a reduction relation, the ARS $\mathcal{A} := (\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow)$ is terminating. Moreover, assume that $\mathfrak{r}$ and $\mathfrak{r}'$ are two $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{r} \Rightarrow \mathfrak{r}'$. By Proposition [Rewrite relation of a TRS], there is a holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{s}$, two $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, and an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\mathfrak{t} \to \mathfrak{t}'$, $\mathfrak{r} = \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t} \cdot \sigma$, and $\mathfrak{r}' = \blacktriangle \cdot \mathfrak{s} \cdot \mathfrak{t}' \cdot \sigma$. From this, and since $\rightsquigarrow$ is $\mathcal{T}$-compatible, we have $\mathfrak{t} \rightsquigarrow \mathfrak{t}'$. Now, since $\rightsquigarrow$ is compatible from factors, by Proposition [Compatibility from factors], the previous properties imply $\mathfrak{r} \rightsquigarrow \mathfrak{r}'$. This shows that the ARS of $\mathcal{T}$ is a sub-ARS of $\mathcal{A}$. By Proposition [Termination witnesses], the desired property holds.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ is terminating, the *reduction relation method* consists in

(1) constructing a binary relation $\rightsquigarrow$ on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}$;

(2) showing that the ARS $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathrm{V}, \rightsquigarrow)$ is terminating;

(3) showing that $\rightsquigarrow$ is compatible from factors;

(4) showing that $\rightsquigarrow$ is $\mathcal{T}$-compatible.

By Theorem [Compatible reduction relations and termination], we obtain the desired property.

In practice, it is difficult to guess such a binary relation $\rightsquigarrow$ which will satisfy the three required properties.

Example

Let us show that the TRS Assoc is terminating by using the reduction relation method.

Let $\mathcal{S}$ (resp. V) be the underlying signature (resp. set of variables) of Assoc.

1. Let for any $v \in \mathsf{V}$ the function lc defined for any $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ by

$$\mathrm{lc}_v \cdot \mathfrak{t} := \sum_{w \in \mathrm{P} \cdot \mathfrak{t}} [\mathrm{w} \cdot \underbracket{\mathfrak{t} \cdot w} \cdot 1 = v] \, 2^{\ell_1 \cdot w}.$$

   For instance, $\mathrm{lc}_1 \cdot \underbracket{m \underbracket{m \underbracket{m12} \underbracket{m \underbracket{m13} 1}} \underbracket{m22}} = 8 + 8 + 1 = 17$.

   Let $\rightsquigarrow$ be the binary relation on $\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that, for any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \rightsquigarrow \mathfrak{t}'$ if for all $v \in \mathsf{V}$, $\mathrm{lc}_v \cdot \mathfrak{t} \geqslant \mathrm{lc}_v \cdot \mathfrak{t}'$ and there exists $v \in \mathsf{V}$ such that $\mathrm{lc}_v \cdot \mathfrak{t} > \mathrm{lc}_v \cdot \mathfrak{t}'$.

2. Let the function $\theta := \mathfrak{t} \mapsto \sum_{v \in \mathsf{V}} \mathrm{lc}_v \cdot \mathfrak{t}$. For any $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ such that $\mathfrak{t} \rightsquigarrow \mathfrak{t}'$, $\theta \cdot \mathfrak{t} > \theta \cdot \mathfrak{t}'$. Since the codomain of θ is $\mathbb{N}$, the ARS $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow)$ is terminating.

3. The compatibility from factors is straightforward (but technical): Properties (CF1) and (CF2) can be proven directly.

4. Let $\mathfrak{t} := m \underbracket{m12} 3$ and $\mathfrak{t}' := m1 \underbracket{m23}$. Since $(\mathrm{lc}_1 \cdot \mathfrak{t}, \mathrm{lc}_2 \cdot \mathfrak{t}, \mathrm{lc}_3 \cdot \mathfrak{t}) = (4, 2, 1)$ and $(\mathrm{lc}_1 \cdot \mathfrak{t}', \mathrm{lc}_2 \cdot \mathfrak{t}', \mathrm{lc}_3 \cdot \mathfrak{t}') = (2, 2, 1)$, $\mathfrak{t} \rightsquigarrow \mathfrak{t}'$. Therefore, $\rightsquigarrow$ is Assoc-compatible.

7.3. Semantic method

Definition

An $\mathcal{S}$*-algebra* is a triple $(X, \mathcal{S}, \mathrm{op})$ where

- $\mathcal{S}$ is a signature, called the *underlying signature*;
- X is a nonempty set, called the *underlying set*;
- op is a function such that for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $\mathrm{op} \cdot c$ is an n-operation on X.

Examples

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ such that $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

- By setting $\mathrm{op} \cdot z := 0$, $\mathrm{op} \cdot s := n \mapsto 1 + n$, and $\mathrm{op} \cdot a := n_1 \mapsto n_2 \mapsto n_1 + n_2$, the triple $(\mathbb{N}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.
- By setting $\mathrm{op} \cdot z := \emptyset$, $\mathrm{op} \cdot s := S \mapsto \mathbb{Z} \setminus S$, and $\mathrm{op} \cdot a := S_1 \mapsto S_2 \mapsto S_1 \cap S_2$, the triple $(\mathcal{P} \cdot \mathbb{Z}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.
- By setting $\mathrm{op} \cdot z := 1$, $\mathrm{op} \cdot s := b \mapsto 1 - b$, and $\mathrm{op} \cdot a := b_1 \mapsto b_2 \mapsto b_1 \times b_2$, the triple $(\{0, 1\}, \mathcal{S}, \mathrm{op})$ is an $\mathcal{S}$-algebra.

Let V be a set of variables and X be a nonempty set.

A V, X*-assignment* is a function $\alpha : \mathsf{V} \to X$.

Given a signature $\mathcal{S}$, an $\mathcal{S}$-algebra $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$, and a V, X-assignment α, the $\mathcal{A}, \alpha$*-evaluation* $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}$ of an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ is defined recursively by

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t} := \begin{cases} \alpha \cdot v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ \mathrm{op} \cdot c \cdot \underbrace{\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_1} \cdot \cdots \cdot \underbrace{\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_n} & \text{otherwise, where } \mathfrak{t} = c \, \mathfrak{t}_1 \dots \mathfrak{t}_n, c \in \mathcal{S} \cdot n, n \in \mathbb{N}, \mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}. \end{cases}$$

Example

Let us consider the first $\mathcal{S}$-algebra $\mathcal{A}$ of the previous example.
Let α be the $\mathsf{V}_{\mathbb{N}}, \mathbb{N}$-assignment α defined by $\alpha \cdot \mathsf{v}_1 := 3$, $\alpha \cdot \mathsf{v}_2 := 1$, and $\alpha \cdot \mathsf{v}_i := 0$, $i \geqslant 3$. We have

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \underbrace{s \, \underbrace{a \, \underbrace{s \mathsf{v}_1} \, \underbrace{a \, s \, \underbrace{s z} \, \mathsf{v}_2}}} = 1 + \underbrace{1 + 3} + \underbrace{1 + \underbrace{1 + 0} + 1} = 8.$$

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X.

The *relation induced by* $\mathcal{R}$ is the binary relation $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}$ such that, for any $\mathcal{S},\mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, $\mathfrak{t} \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{t}'$ holds if $\mathrm{ev}_{\mathcal{A},\alpha}\cdot\mathfrak{t} \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A},\alpha}\cdot\mathfrak{t}'$ for all V, X-assignments α.

Examples

Let us consider the $\mathcal{S}$-algebra $\mathcal{A}$ of the previous example.

Let $>$ be the usual ‘‘greater-than’’ relation on $\mathbb{N}$.

- ☐ We have

$$a \mathsf{v}_1 \lfloor s \mathsf{v}_2 \rfloor \rightsquigarrow_{\mathcal{A},>} a \mathsf{v}_1 \mathsf{v}_2.$$

  Indeed, for any $\mathsf{V}_{\mathbb{N}}, \mathbb{N}$ assignment α, we have

$$\mathrm{ev}_{\mathcal{A},\alpha}\cdot\lfloor a \mathsf{v}_1 \lfloor s \mathsf{v}_2 \rfloor \rfloor = \lfloor \alpha\cdot\mathsf{v}_1 \rfloor + \lfloor \alpha\cdot\mathsf{v}_2 \rfloor + 1 > \lfloor \alpha\cdot\mathsf{v}_1 \rfloor + \lfloor \alpha\cdot\mathsf{v}_2 \rfloor = \mathrm{ev}_{\mathcal{A},\alpha}\cdot\lfloor a \mathsf{v}_1 \mathsf{v}_2 \rfloor.$$

- ☐ By setting $\mathfrak{t} := a\mathsf{v}_1\mathsf{v}_2$ and $\mathfrak{t}' := s\mathsf{v}_1$, we have neither $\mathfrak{t} \rightsquigarrow_{\mathcal{A},>} \mathfrak{t}'$ nor $\mathfrak{t}' \rightsquigarrow_{\mathcal{A},>} \mathfrak{t}$.

Let $\mathcal{S}$ be a signature, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X.

The $\mathcal{S}$-algebra $\mathcal{A}$ is *$\mathcal{R}$-monotone* if for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, and $x_1, \dots, x_i, x_i', \dots, x_n \in X$,

$$x_i \, \mathcal{R} \, x_i' \quad \text{implies} \quad \mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_i \cdot \cdots \cdot x_n \;\mathcal{R}\; \mathrm{op} \cdot c \cdot x_1 \cdot \cdots \cdot x_i' \cdot \cdots \cdot x_n.$$

Example

Let us consider the $\mathcal{S}$-algebra $\mathcal{A}$ and the binary relation $>$ of the previous example. We immediately have that $\mathcal{A}$ is $>$-monotone.

Theorem [Reduction relations from $\mathcal{S}$-algebras]

Let $\mathcal{S}$ be a signature, $\mathcal{A} := (X, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-algebra, and $\mathcal{R}$ be a binary relation on X. If the ARS $(X, \mathcal{R})$ is terminating and $\mathcal{A}$ is $\mathcal{R}$-monotone, then $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$ is a reduction relation.

This result is due to [Z. Manna, S. Ness, On the termination of Markov algorithms, 1970].

Proof (of Theorem [Reduction relations from $\mathcal{S}$-algebras]). Since X is nonempty, pick $x \in X$ and let α be the V, X-assignment defined by $\alpha \cdot v := x$ for any $v \in \mathsf{V}$. Let $(u_i)_{i \in \mathbb{N}}$ be an infinite rewrite sequence in $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow_{\mathcal{A},\mathcal{R}})$. Hence, for any $i \in \mathbb{N}$, $u_i \rightsquigarrow_{\mathcal{A},\mathcal{R}} u_{i+1}$, so that, by definition of $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$, for any $i \in \mathbb{N}$, $\mathrm{ev}_{\mathcal{A},\alpha} \cdot u_i \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A},\alpha} \cdot u_{i+1}$. Therefore, $(\mathrm{ev}_{\mathcal{A},\alpha} \cdot u_i)_{i \in \mathbb{N}}$ is an infinite rewrite sequence in $(X, \mathcal{R})$. Since $(X, \mathcal{R})$ is terminating, such an infinite rewrite sequence $(u_i)_{i \in \mathbb{N}}$ cannot exists in $(\mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}, \rightsquigarrow_{\mathcal{A},\mathcal{R}})$. Hence, this ARS is terminating.

Let $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, and $\mathfrak{t}_1, \dots, \mathfrak{t}_i, \mathfrak{t}'_i, \dots, \mathfrak{t}_n \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathfrak{t}_i \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{t}'_i$. By definition of $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$, for any V, X-assignment α, $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_i \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'_i$. Since $\mathcal{A}$ is $\mathcal{R}$-monotone, by setting $\mathfrak{s} := c \, \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}_i \cdot \cdots \cdot \mathfrak{t}_n$ and $\mathfrak{s}' := c \, \mathfrak{t}_1 \cdot \cdots \cdot \mathfrak{t}'_i \cdot \cdots \cdot \mathfrak{t}_n$, we have $\mathfrak{s} \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{s}'$ because

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{s} = \mathrm{op} \cdot c \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_1 \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_i \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_n \rfloor \; \mathcal{R} \; \mathrm{op} \cdot c \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_1 \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}'_i \rfloor \cdot \cdots \cdot \lfloor \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{t}_n \rfloor = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{s}'.$$

Given a V, X-assignment α and an $\mathcal{S}, \mathsf{V}$-substitution σ, let $\alpha \circ \sigma$ be the V, X-assignment defined, for any $v \in \mathsf{V}$, by $\lfloor \alpha \circ \sigma \rfloor \cdot v := \mathrm{ev}_{\mathcal{A},\alpha} \cdot \lfloor \sigma \cdot v \rfloor$. It follows, by structural induction on an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$ that $\mathrm{ev}_{\mathcal{A},\alpha} \cdot \lfloor \overline{\sigma} \cdot \mathfrak{t} \rfloor = \mathrm{ev}_{\mathcal{A},\alpha \circ \sigma} \cdot \mathfrak{t}$. Now, let $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V}$ such that $\mathfrak{t} \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{t}'$. From this last assumption, for any $\mathcal{S}, \mathsf{V}$-substitution σ, by setting $\mathfrak{s} := \overline{\sigma} \cdot \mathfrak{t}$ and $\mathfrak{s}' := \overline{\sigma} \cdot \mathfrak{t}'$, we have $\mathfrak{s} \rightsquigarrow_{\mathcal{A},\mathcal{R}} \mathfrak{s}'$ because

$$\mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{s} = \mathrm{ev}_{\mathcal{A},\alpha \circ \sigma} \cdot \mathfrak{t} \; \mathcal{R} \; \mathrm{ev}_{\mathcal{A},\alpha \circ \sigma} \cdot \mathfrak{t}' = \mathrm{ev}_{\mathcal{A},\alpha} \cdot \mathfrak{s}'.$$

This shows that $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ is compatible from factors and implies the statement of the theorem.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ is terminating, the *semantic method* consists in

(1) constructing an $\mathcal{S}$-algebra $(X, \mathcal{S}, \mathrm{op})$;

(2) constructing a binary relation $\mathcal{R}$ on X;

(3) showing that the ARS $(X, \mathcal{R})$ is terminating;

(4) showing that $\mathcal{A}$ is $\mathcal{R}$-monotone;

(5) showing that $\rightsquigarrow_{\mathcal{A},\mathcal{R}}$ is $\mathcal{T}$-compatible.

By Theorem [Reduction relations from $\mathcal{S}$-algebras], and Theorem [Compatible reduction relations and termination], we obtain the desired property.

In practice, it is difficult to guess such an $\mathcal{S}$-algebra $\mathcal{A}$ and such a binary relation $\mathcal{R}$ which will satisfy the three required properties.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}_{\mathbb{N}}, \rightarrow)$ be the TRS such that $\mathcal{S} := (\{z, s, f\}, \mathrm{ar})$ where $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot f = 1$, and $f \llcorner s \mathsf{v}_1 \lrcorner \rightarrow f \llcorner f \mathsf{v}_1 \lrcorner$. Let us show that $\mathcal{T}$ is terminating by using the semantic method.

1. Let the $\mathcal{S}$-algebra $\mathcal{A} := \left(\mathbb{N}^2, \mathcal{S}, \mathrm{op}\right)$ such that $\mathrm{op} \cdot z := (0, 0)$, $\mathrm{op} \cdot s := (n_1, n_2) \mapsto (n_1 + 1, n_2)$, and $\mathrm{op} \cdot f := (n_1, n_2) \mapsto (n_1, n_2 + 1)$.
2. Let the binary relation $\mathcal{R}$ on $\mathbb{N}^2$ defined by $(i_1, i_2) \, \mathcal{R} \, (i'_1, i'_2)$ if $i_1 > i'_1$, or $i_1 = i'_1$ and $i_2 > i'_2$.
3. It is immediate that $\left(\mathbb{N}^2, \mathcal{R}\right)$ is terminating.
4. It is immediate that $\mathcal{A}$ is $\mathcal{R}$-monotone.
5. For any $\mathsf{V}_{\mathbb{N}}, \mathbb{N}^2$-assignment α such that $\alpha \cdot \mathsf{v}_1 = (i_1, i_2)$, we have

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \llcorner f \llcorner s \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot s \cdot \llcorner \alpha \cdot \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot s \cdot (i_1, i_2) \lrcorner = \mathrm{op} \cdot f \cdot (i_1 + 1, i_2) = (i_1 + 1, i_2 + 1)$$

and

$$\mathrm{ev}_{\mathcal{A}, \alpha} \cdot \llcorner f \llcorner f \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot f \cdot \llcorner \alpha \cdot \mathsf{v}_1 \lrcorner \lrcorner = \mathrm{op} \cdot f \cdot \llcorner \mathrm{op} \cdot f \cdot (i_1, i_2) \lrcorner = \mathrm{op} \cdot f \cdot (i_1, i_2 + 1) = (i_1, i_2 + 2).$$

Therefore, as $(i_1 + 1, i_2 + 1) \, \mathcal{R} \, (i_1, i_2 + 2)$, we have $f \llcorner s \mathsf{v}_1 \lrcorner \rightsquigarrow_{\mathcal{A}, \mathcal{R}} f \llcorner f \mathsf{v}_1 \lrcorner$. This shows that $\rightsquigarrow_{\mathcal{A}, \mathcal{R}}$ is $\mathcal{T}$-compatible.

7.4. Polynomial interpretations

Let X be the set of variables $\{\mathsf{x}_i : i \in \mathbb{N} \setminus \{0\}\}$ and X_n be the subset $\{\mathsf{x}_1, \dots, \mathsf{x}_n\}$ of X for any $n \in \mathbb{N}$.

Definition

Let $\mathcal{S}$ be a signature. An *$\mathcal{S}$-polynomial interpretation* is an $\mathcal{S}$-algebra $(N, \mathcal{S}, \mathrm{op})$ where

- N is a subset of $\mathbb{N} \setminus \{0\}$;
- for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, the function $\mathrm{op} \cdot c$ is functionally equivalent to a function $\mathsf{x}_1 \mapsto \dots \mapsto \mathsf{x}_n \mapsto f_c$ such that f_c is a $\mathbb{K}, \mathrm{Mon} \cdot \mathsf{X}_n$-polynomial where coefficients belong to $\mathbb{N}$.

Example

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ where $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

The triple $(N, \mathcal{S}, \mathrm{op})$ where $\mathrm{op} \cdot z := 10$, $\mathrm{op} \cdot s := n_1 \mapsto n_1^2$, $\mathrm{op} \cdot a := n_1 \mapsto n_2 \mapsto 3n_1 + 2$, and $N := \{2n : n \in \mathbb{N} \setminus \{0\}\}$ is an $\mathcal{S}$-polynomial interpretation.

Indeed, $\mathrm{op} \cdot z$ is functionally equivalent to the constant function 10, $\mathrm{op} \cdot s$ is functionally equivalent to the function $\mathsf{x}_1 \mapsto \mathsf{x}_1^2$, and $\mathrm{op} \cdot a$ is functionally equivalent to the function $\mathsf{x}_1 \mapsto \mathsf{x}_2 \mapsto 3\mathsf{x}_1 + 2$.

We have $f_z = 10$, $f_s = \mathsf{x}_1^2$, and $f_a = 3\mathsf{x}_1 + 2$.

Let V be a set of variables and Z_{V} be the set of variables $\{\mathsf{z}_v : v \in \mathsf{V}\}$.

If f is a $\mathbb{K}, \mathrm{Mon}\cdot\mathsf{X}_n$-polynomial, $n \in \mathbb{N}$, and $f'_1 \dots f'_n$ is a sequence of $\mathbb{K}, \mathrm{Mon}\cdot\mathsf{Z}_{\mathsf{V}}$-polynomials, let $f(f'_1, \dots, f'_n)$ be the $\mathbb{K}, \mathrm{Mon}\cdot\mathsf{Z}_{\mathsf{V}}$-polynomial obtained by substituting by f'_i each occurrence of a variable x_i in f, for all $i \in [n]$.

Let $\mathcal{S}$ be a signature and $(N, \mathcal{S}, \mathrm{op})$ be an $\mathcal{S}$-polynomial interpretation.

Given an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{t}$, the *polynomial* of $\mathfrak{t}$ is the $\mathbb{K}, \mathrm{Mon}\cdot\mathsf{Z}_{\mathsf{V}}$-polynomial defined recursively by

$$\mathsf{Pol}\cdot\mathfrak{t} := \begin{cases} \mathsf{z}_v & \text{if } \mathfrak{t} = v \text{ for a } v \in \mathsf{V}, \\ f_c(\mathsf{Pol}\cdot\mathfrak{t}_1, \dots, \mathsf{Pol}\cdot\mathfrak{t}_n) & \text{otherwise, where } \mathfrak{t} = c\,\mathfrak{t}_1 \dots \mathfrak{t}_n, c \in \mathcal{S}\cdot n, n \in \mathbb{N}, \mathfrak{t}_1, \dots, \mathfrak{t}_n \in \mathfrak{T}\cdot\mathcal{S}\cdot\mathsf{V}. \end{cases}$$

Example

Let $\mathcal{S}$ be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ whose underlying set is $\{\mathsf{c}_0, \mathsf{c}_2, \mathsf{c}_3\}$ and the $\mathcal{S}$-polynomial interpretation $(N, \mathcal{S}, \mathrm{op})$ such that $N := \mathbb{N} \setminus \{0\}$, $\mathrm{op}\cdot\mathsf{c}_0 := 2$, $\mathrm{op}\cdot\mathsf{c}_2 := n_1 \mapsto n_2 \mapsto n_1^2 + n_2$, and $\mathrm{op}\cdot\mathsf{c}_3 := n_1 \mapsto n_2 \mapsto n_3 \mapsto n_1 n_2 + n_2 n_3$. We have $f_{\mathsf{c}_0} = 2$, $f_{\mathsf{c}_2} = \mathsf{x}_1^2 + \mathsf{x}_2$, and $f_{\mathsf{c}_3} = \mathsf{x}_1\mathsf{x}_2 + \mathsf{x}_2\mathsf{x}_3$.
Moreover,

$$\mathsf{Pol}\cdot\mathsf{c}_3\,\mathsf{c}_3\,\mathsf{v}_1\,\mathsf{v}_2\,\mathsf{v}_1\,\mathsf{c}_2\,\mathsf{v}_3\,\mathsf{v}_5\,\mathsf{c}_0 = f_{\mathsf{c}_3}(f_{\mathsf{c}_3}(\mathsf{z}_{\mathsf{v}_1}, \mathsf{z}_{\mathsf{v}_2}, \mathsf{z}_{\mathsf{v}_1}), f_{\mathsf{c}_2}(\mathsf{z}_{\mathsf{v}_3}, \mathsf{z}_{\mathsf{v}_5}), f_{\mathsf{c}_0})$$

$$= f_{\mathsf{c}_3}\left(2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}, \mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}, 2\right) = 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\left(\mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}\right) + 2\left(\mathsf{z}_{\mathsf{v}_3}^2 + \mathsf{z}_{\mathsf{v}_5}\right) = 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\mathsf{z}_{\mathsf{v}_3}^2 + 2\mathsf{z}_{\mathsf{v}_1}\mathsf{z}_{\mathsf{v}_2}\mathsf{z}_{\mathsf{v}_5} + 2\mathsf{z}_{\mathsf{v}_3}^2 + 2\mathsf{z}_{\mathsf{v}_5}.$$