

Let \mathcal{S} be a signature and $\mathcal{A} := (N, \mathcal{S}, \text{op})$ be an \mathcal{S} -polynomial interpretation.

When, for any $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, there is a monomial $x_1^{e_1} \dots x_i^{e_i} \dots x_n^{e_n} \in \text{Supp} \cdot f_c$ such that $e_i \geq 1$, \mathcal{A} is *full*.

Example

The polynomial interpretation of the first example of this part is not full. Besides, by considering the signature \mathcal{S} and the set N of this example, the triple $(N, \mathcal{S}, \text{op})$ where $\text{op} \cdot z := 1$, $\text{op} \cdot s := n_1 \mapsto n_1$, and $\text{op} \cdot a := n_1 \mapsto n_2 \mapsto n_1 + n_2^2 + 3n_1n_2$ is a full \mathcal{S} -polynomial interpretation.

Proposition [Monotonicity of full polynomial interpretations]

Let \mathcal{S} be a signature and \mathcal{A} be an \mathcal{S} -polynomial interpretation. If \mathcal{A} is full, then \mathcal{A} is $>$ -monotone, where $>$ is the ‘greater-than’ relation on integers restricted to the underlying set of \mathcal{A} .

Proof. Assume that $\mathcal{A} = (N, \mathcal{S}, \text{op})$. Let $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $i \in [n]$, and $m_1, \dots, m_i, m'_i, \dots, m_n \in N$ such that $m_i > m'_i$. Since \mathcal{A} is full, there is a monomial $x_1^{e_1} \dots x_i^{e_i} \dots x_n^{e_n}$ in the support of f_c with $e_i \geq 1$. Since for all $j \in [n] \setminus \{i\}$, $m_j \neq 0$, the inequality $m_i > m'_i$ implies $m_1^{e_1} \dots m_i^{e_i} \dots m_n^{e_n} > m_1^{e_1} \dots m'_i^{e_i} \dots m_n^{e_n}$. Therefore, and since each coefficient of f_c belongs to \mathbb{N} , $f_c(m_1, \dots, m_i, \dots, m_n) > f_c(m_1, \dots, m'_i, \dots, m_n)$ so that $\text{op} \cdot c \cdot m_1 \cdot \dots \cdot m_i \cdot \dots \cdot m_n > \text{op} \cdot c \cdot m_1 \cdot \dots \cdot m'_i \cdot \dots \cdot m_n$. This shows the desired property.

In order to prove that a TRS $\mathcal{T} := (\mathcal{S}, \mathcal{V}, \rightarrow)$ is terminating, the *polynomial interpretation method*, due to [D. Lankford, Canonical algebraic simplification in computational logic, 1975], consists in

- (1) constructing an \mathcal{S} -polynomial interpretation $\mathcal{A} := (N, \mathcal{S}, \text{op})$;
- (2) showing that \mathcal{A} is full;
- (3) showing that for any rewrite rule (t, t') of \mathcal{T} , the $\mathbb{K}, \text{Mon}\cdot\mathbb{Z}_{\mathcal{V}}$ -polynomial $\text{Pol}\cdot t - \text{Pol}\cdot t'$ is positive for any evaluation on N .

By Proposition [Monotonicity of full polynomial interpretations], Theorem [Reduction relation from \mathcal{S} -algebras], and Theorem [Compatible reduction relations and termination], we obtain the desired property.

At Step (1), we must be careful in the choice of N so that for any $c \in \mathcal{S}\cdot n$, $n \in \mathbb{N}$, all evaluations of f_c on N belong to N .

Step (3) amounts to show that for any rewrite rule (t, t') of \mathcal{T} and any \mathcal{V}, N -assignment α , $\text{ev}_{\mathcal{A}, \alpha}\cdot t > \text{ev}_{\mathcal{A}, \alpha}\cdot t'$.

Note that this decision problem is in general undecidable, as a consequence of the **Hilbert's Tenth Problem**.

Example

Let us prove the termination of the TRS Assoc_2 by the polynomial interpretation method.

Let \mathcal{S} be the underlying signature of Assoc_2 .

1. Let $\mathcal{A} := (N, \mathcal{S}, \text{op})$ be the \mathcal{S} -polynomial interpretation such that $N := \mathbb{N} \setminus \{0\}$, $\text{op} \cdot m := n_1 \mapsto n_2 \mapsto 2n_1 + n_2$, and $\text{op} \cdot m' := n_1 \mapsto n_2 \mapsto n_1 + 2n_2$.
2. It is immediate that \mathcal{A} is full.
3. We have

$$\text{Pol} \cdot \underline{m(m12)3} = 2(2z_1 + z_2) + z_3 = 4z_1 + 2z_2 + z_3$$

and

$$\text{Pol} \cdot \underline{m1(m23)} = 2z_1 + (2z_2 + z_3) = 2z_1 + 2z_2 + z_3.$$

Since $\text{Pol} \cdot \underline{m(m12)3} - \text{Pol} \cdot \underline{m1(m23)} = 2z_1$, this $\mathbb{K}, \text{Mon} \cdot \mathbb{Z}_{\{1,2,3\}}$ -polynomial is positive on N .

In the same way, we have $\text{Pol} \cdot \underline{m'1(m'23)} - \text{Pol} \cdot \underline{m'(m'12)3} = 2z_3$, so that the same property holds.

Exercise ○○○○

Let the signature $\mathcal{S} := (\{f, m\}, \text{ar})$ such that $\text{ar}\cdot f = 1$ and $\text{ar}\cdot m = 2$.

Let the TRS $\mathcal{T} := (\mathcal{S}, V_{\mathbb{N}}, \rightarrow)$ such that $m_{\lfloor f v_1 \rfloor \lfloor f v_2 \rfloor} \rightarrow f_{\lfloor m v_1 v_2 \rfloor}$ and $m_{\lfloor m v_1 v_2 \rfloor v_3} \rightarrow m v_1 \lfloor m v_2 v_3 \rfloor$.

By using the polynomial interpretation method, show that \mathcal{T} is terminating.

Exercise ○○○○

By using the polynomial interpretation method, show that the TRS `NatAdd` is terminating.

Exercise ○○○○

Let V be the set of variables $\mathbb{N} \setminus \{0\}$ and \mathcal{S} be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{c_2, c_3\}$.

Let the TRS $\text{Motz} := (\mathcal{S}, V, \rightarrow)$ such that \rightarrow satisfies

$$\begin{aligned} c_2 \lfloor c_2 12 \rfloor 3 &\rightarrow c_2 1 \lfloor c_2 23 \rfloor, & c_3 \lfloor c_2 12 \rfloor 34 &\rightarrow c_2 1 \lfloor c_3 234 \rfloor, \\ c_2 \lfloor c_3 123 \rfloor 4 &\rightarrow c_3 12 \lfloor c_2 34 \rfloor, & c_3 \lfloor c_3 123 \rfloor 45 &\rightarrow c_3 12 \lfloor c_3 45 \rfloor, \end{aligned}$$

By using the polynomial interpretation method, show that Motz is terminating.

Exercise ○○○○

By using the polynomial interpretation method, show that the TRS `Schr` is terminating.

/ Termination

7.5. Syntactic method

Definition

Let \mathcal{S} be a signature and V be a set of variables. A binary relation \rightsquigarrow on $\mathcal{T}\cdot\mathcal{S}\cdot V$ is a *simplification relation* if

- \rightsquigarrow is acyclic;
- \rightsquigarrow is compatible from factors;
- for any \mathcal{S}, V -term t and any position $w \in P\cdot t \setminus \{\epsilon\}$, $t \rightsquigarrow t.w$.

Theorem [Simplification relations and termination]

For any finite signature \mathcal{S} and any finite set of variables V , if \rightsquigarrow is a simplification relation on $\mathcal{T}\cdot\mathcal{S}\cdot V$, then the ARS $(\mathcal{T}\cdot\mathcal{S}\cdot V, \rightsquigarrow)$ is terminating.

Theorem [Simplification relations and termination] implies that any simplification relation is a **reduction relation**. The **converse** of this property is **false**.

Exercise ○○○○

Give an example of a finite signature \mathcal{S} , of a finite set of variables V , and of a reduction relation \rightsquigarrow on $\mathcal{T}\cdot\mathcal{S}\cdot V$ such that \rightsquigarrow is nonempty and \rightsquigarrow is not a simplification relation.

Let $\mathcal{S} := (C, \text{ar})$ be a signature, V be a set of variables, and \succ be a binary relation on C .

An \mathcal{S}, V, \succ -weight function is a function $\omega : C \sqcup V \rightarrow \mathbb{R}_{\geq 0}$ such that

1. all variables of V are sent via ω to the same nonnegative real number ω_V ;
2. for all $c \in \mathcal{S} \cdot 0$, $\omega \cdot c \geq \omega_V$;
3. for all $c \in \mathcal{S} \cdot 1$, $\omega \cdot c = 0$ implies that for any $c' \in C \setminus \{c\}$, $c \succ c'$.

The ω -weight of a \mathcal{S}, V -term t is the nonnegative real number

$$\omega \cdot t := \sum_{i \in [l \cdot t]} \omega \cdot \underline{w \cdot t \cdot i}.$$

Example

Let \mathcal{S} be the sub-graded set of $\mathcal{S}_{\mathbb{N}^2}$ consisting of $\{c_0, c_1, c_2, c_4\}$. Let \succ be the binary relation on the underlying set of \mathcal{S} satisfying $c_1 \succ c_0$, $c_1 \succ c_2$, $c_1 \succ c_4$, and $c_0 \succ c_2$.

Let the $\mathcal{S}, V_{\mathbb{N}}, \succ$ -weight function ω defined by $\omega \cdot v := 1$ for any $v \in V_{\mathbb{N}}$, $\omega \cdot c_0 := \frac{3}{2}$, $\omega \cdot c_1 := 0$, $\omega \cdot c_2 := \frac{1}{2}$, $\omega \cdot c_4 := 2$.

We have

$$\omega \cdot c_4 c_0 c_0 \underline{c_2 v_2 v_1} \underline{c_1 v_2} = 2 + \frac{3}{2} + \frac{3}{2} + \frac{1}{2} + 1 + 1 + \frac{1}{2} + 1 = 9.$$

Let \mathcal{S} be a signature, \mathcal{V} be a set of variables, \succ be a binary relation on the underlying set of \mathcal{S} , and ω be an $\mathcal{S}, \mathcal{V}, \succ$ -weight function.

The \succ, ω -Knuth-Bendix relation is the binary relation $\succ_{\text{KB}}^{\succ, \omega}$ on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathcal{V}$ defined recursively as follows. For any \mathcal{S}, \mathcal{V} -terms t and t' , we have $t \succ_{\text{KB}}^{\succ, \omega} t'$ if for all $v \in \mathcal{V}$, $l_v \cdot t \geq l_v \cdot t'$ and one of the following assertions holds:

- [Weight Case] $\omega \cdot t > \omega \cdot t'$;
- [Unary Case] $\omega \cdot t = \omega \cdot t'$, $t = c \underline{c \dots [cv] \dots}$ with $c \in \mathcal{S} \cdot 1$, $v \in \mathcal{V}$, $l_c \cdot t = n$, $n \geq 1$, and $t' = v$;
- [Precedence Case] $\omega \cdot t = \omega \cdot t'$, $t = c \underline{[t \cdot 1] \dots [t \cdot n]}$ with $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, $t' = c' \underline{[t' \cdot 1] \dots [t' \cdot n']}$ with $c' \in \mathcal{S} \cdot n'$, $n' \in \mathbb{N}$, and $c \succ c'$;
- [Lexicographic Case] $\omega \cdot t = \omega \cdot t'$, $t = c \underline{[t \cdot 1] \dots [t \cdot n]}$, $t' = c \underline{[t' \cdot 1] \dots [t' \cdot n]}$ with $c \in \mathcal{S} \cdot n$, $n \in \mathbb{N}$, and there is $i \in [n]$ such that $t \cdot i \succ_{\text{KB}}^{\succ, \omega} t' \cdot i$ and for any $j \in [i - 1]$, $t \cdot j = t' \cdot j$.

Proposition [Knuth-Bendix simplification relation]

For any signature \mathcal{S} , any set of variables \mathcal{V} , any binary relation \succ on \mathcal{S} such that the ARS (\mathcal{S}, \succ) is terminating, and any $\mathcal{S}, \mathcal{V}, \succ$ -weight function ω , the binary relation $\succ_{\text{KB}}^{\succ, \omega}$ is a simplification relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\mathcal{V}$.

This result is due to [D. Knuth, P. Bendix, Simple Word Problems in Universal Algebras, 1970] and [N. Dershowitz, Orderings for Term-Rewriting Systems, 1982].