Examples

Let the signature $\mathcal{S} := (\{z, s, s'.a\}, \mathrm{ar})$ such that $\mathrm{ar}{\cdot}z = 0$, $\mathrm{ar}{\cdot}s = 1$, $\mathrm{ar}{\cdot}s' = 1$, and $\mathrm{ar}{\cdot}a = 2$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $a \succ s$, $s \succ z$, $s' \succ z$, $s' \succ s$, and $s' \succ a$.

Let ω be the $\mathcal{S}, \mathsf{V}, \succ$-weight function defined by $\omega_{\mathsf{V}} := 1$, $\omega{\cdot}z := 1$, $\omega{\cdot}s := 1$, $\omega{\cdot}s' := 0$, and $\omega{\cdot}a = 2$.

Let the $\mathcal{S}, \mathsf{V}_{\mathbb{N}}$-term

$$\mathrm{t} := a \lfloor a z \mathsf{v}_1 \rfloor \lfloor s \lfloor a \mathsf{v}_2 \lfloor s z \rfloor \rfloor \rfloor$$

- Let $\mathrm{t}' := a \mathsf{v}_1 \mathsf{v}_2$. We have $\mathrm{t} >_{\mathrm{KB}}^{\succ, \omega} \mathrm{t}'$ by applying the Weight Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathrm{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathrm{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathrm{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_2}{\cdot}\mathrm{t}'$, and $\omega{\cdot}\mathrm{t} = 12 > 4 = \omega{\cdot}\mathrm{t}'$.
- Let $\mathrm{t}' := s \lfloor a \lfloor a \lfloor s z \rfloor \mathsf{v}_1 \rfloor \lfloor a z z \rfloor \rfloor$. We have $\mathrm{t} >_{\mathrm{KB}}^{\succ, \omega} \mathrm{t}'$ by applying the Precedence Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathrm{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathrm{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathrm{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}{\cdot}\mathrm{t}'$, $\omega{\cdot}\mathrm{t} = 12 = \omega{\cdot}\mathrm{t}'$, and $a \succ s$.
- Let $\mathrm{t}' := a \lfloor a z \mathsf{v}_1 \rfloor \lfloor s \lfloor s \lfloor a z z \rfloor \rfloor \rfloor$. We have $\mathrm{t} >_{\mathrm{KB}}^{\succ, \omega} \mathrm{t}'$ by applying the Lexicographic Case: we have $\ell_{\mathsf{v}_1}{\cdot}\mathrm{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathrm{t}'$, $\ell_{\mathsf{v}_2}{\cdot}\mathrm{t} = 1 \geqslant 0 = \ell_{\mathsf{v}_2}{\cdot}\mathrm{t}'$, $\omega{\cdot}\mathrm{t} = 12 = \omega{\cdot}\mathrm{t}'$, $\mathrm{t}{\cdot}1 = \mathrm{t}'{\cdot}1$, and $\mathrm{t}{\cdot}1 >_{\mathrm{KB}}^{\succ, \omega} \mathrm{t}'{\cdot}1$.

Moreover, by setting $\mathrm{t} := s' \lfloor s' \lfloor s' \mathsf{v}_1 \rfloor \rfloor$ and $\mathrm{t}' := \mathsf{v}_1$, we have $\mathrm{t} >_{\mathrm{KB}}^{\succ, \omega} \mathrm{t}'$ by applying the Unary Case: $\ell_{\mathsf{v}_1}{\cdot}\mathrm{t} = 1 \geqslant 1 = \ell_{\mathsf{v}_1}{\cdot}\mathrm{t}'$, $\ell_s{\cdot}\mathrm{t} \geqslant 1$, and $\omega{\cdot}\mathrm{t} = 0 = \omega{\cdot}\mathrm{t}'$.

Example

Let $\mathcal{T} := (\mathcal{S}, \{\mathsf{v}\}, \to)$ be the TRS such that $\mathcal{S} := (\{f, g\}, \mathrm{ar})$ where $\mathrm{ar} \cdot f = 1$ and $\mathrm{ar} \cdot g = 1$, and $g \llcorner g\mathsf{v} \lrcorner \to f\mathsf{v}$ and $f \llcorner g\mathsf{v} \lrcorner \to g \llcorner f\mathsf{v} \lrcorner$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $f \succ g$.

Let ω be the $\mathcal{S}, \{\mathsf{v}\}, \succ$-weight function defined by $\omega \cdot \mathsf{v} := 1$, $\omega \cdot f := 1$, and $\omega \cdot g := 1$.

It can be easily checked that $g \llcorner g\mathsf{v} \lrcorner >_{\mathrm{KB}}^{\succ, \omega} f\mathsf{v}$ and $f \llcorner g\mathsf{v} \lrcorner >_{\mathrm{KB}}^{\succ, \omega} g \llcorner f\mathsf{v} \lrcorner$. Therefore, $\mathcal{T}$ is terminating.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ be the TRS such that $\mathcal{S} := (\{s, a\}, \mathrm{ar})$ where $\mathrm{ar} \cdot s = 1$ and $\mathrm{ar} \cdot a = 2$, $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4\}$, and $a \llcorner s\mathsf{v}_1 \lrcorner \llcorner a\mathsf{v}_2\mathsf{v}_3 \lrcorner \to a\mathsf{v}_1 \llcorner a \llcorner s \llcorner s\mathsf{v}_2 \lrcorner \lrcorner \mathsf{v}_3 \lrcorner$ and $a \llcorner s\mathsf{v}_1 \lrcorner \llcorner a\mathsf{v}_2 \llcorner a\mathsf{v}_3\mathsf{v}_4 \lrcorner \lrcorner \to a\mathsf{v}_1 \llcorner a\mathsf{v}_3 \llcorner a\mathsf{v}_2\mathsf{v}_4 \lrcorner \lrcorner$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $s \succ a$.

Let ω be the $\mathcal{S}, \mathsf{V}, \succ$-weight function defined by $\omega_{\mathsf{v}} := 1$, $\omega \cdot s := 0$, and $\omega \cdot a := 0$.

It can be easily checked that $a \llcorner s\mathsf{v}_1 \lrcorner \llcorner a\mathsf{v}_2\mathsf{v}_3 \lrcorner >_{\mathrm{KB}}^{\succ, \omega} a\mathsf{v}_1 \llcorner a \llcorner s \llcorner s\mathsf{v}_2 \lrcorner \lrcorner \mathsf{v}_3 \lrcorner$ and $a \llcorner s\mathsf{v}_1 \lrcorner \llcorner a\mathsf{v}_2 \llcorner a\mathsf{v}_3\mathsf{v}_4 \lrcorner \lrcorner >_{\mathrm{KB}}^{\succ, \omega} a\mathsf{v}_1 \llcorner a\mathsf{v}_3 \llcorner a\mathsf{v}_2\mathsf{v}_4 \lrcorner \lrcorner$. Therefore, $\mathcal{T}$ is terminating.

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\succ$ be a binary relation on the underlying set of $\mathcal{S}$.

The *$\succ$-lexicographic path relation* is the binary relation $>_{\mathrm{LP}}^{\succ}$ on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$ defined recursively as follows. For any $\mathcal{S},\mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$, we have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ if one of the following assertion holds:

- [Subterm Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, and there exists $i \in [n]$ such that $\mathfrak{t}{\cdot}i = \mathfrak{t}'$ or $\mathfrak{t}{\cdot}i >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$;
- [Precedence Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $\mathfrak{t}' = c' \llcorner \mathfrak{t}'{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}'{\cdot}n' \lrcorner$ with $c' \in \mathcal{S}{\cdot}n'$, $n' \in \mathbb{N}$, $c \succ c'$, and $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}i'$ for any $i' \in [n']$;
- [Lexicographic Case] $\mathfrak{t} = c \llcorner \mathfrak{t}{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}{\cdot}n \lrcorner$, $\mathfrak{t}' = c \llcorner \mathfrak{t}'{\cdot}1 \lrcorner \dots \llcorner \mathfrak{t}'{\cdot}n \lrcorner$ with $c \in \mathcal{S}{\cdot}n$, $n \in \mathbb{N}$, $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}i$ for all $i \in [n]$, and there is $j \in [n]$ such that $\mathfrak{t}{\cdot}j >_{\mathrm{LP}}^{\succ} \mathfrak{t}'{\cdot}j$ and for any $j' \in [i-1]$, $\mathfrak{t}{\cdot}j' = \mathfrak{t}'{\cdot}j'$.

Proposition [Lexicographic path simplification relation]

For any signature $\mathcal{S}$, any set of variables V, and any binary relation $\succ$ on $\mathcal{S}$ such that the ARS $(\mathcal{S}, \succ)$ is terminating, the binary relation $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

This result is due to [S. Kamin, J.-J. Levy, Two Generalizations of the Recursive Path Ordering, 1980].

Examples

Let the signature $\mathcal{S} := (\{z, s, a\}, \mathrm{ar})$ such that $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot a = 2$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $a \succ s$ and $s \succ z$.

Let the $\mathcal{S}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t} := a \lfloor s\mathsf{v}_1 \rfloor \lfloor a z \lfloor s\mathsf{v}_2 \rfloor \rfloor$.

- Let $\mathfrak{t}' := s\mathsf{v}_2$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying recursively the Subterm Case: we have $\mathfrak{t} \cdot 22 = \mathfrak{t}'$, which implies $\mathfrak{t} \cdot 2 >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$, which implies $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$.
- Let $\mathfrak{t}' := sz$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying the Precedence Case: we have $a \succ s$ and $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 1$.
- Let $\mathfrak{t}' := a \lfloor s\mathsf{v}_1 \rfloor \lfloor s\mathsf{v}_2 \rfloor$. We have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}'$ by applying the Lexicographic Case: we have $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 1$, $\mathfrak{t} >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 2$, $\mathfrak{t} \cdot 1 = \mathfrak{t}' \cdot 1$, and $\mathfrak{t} \cdot 2 >_{\mathrm{LP}}^{\succ} \mathfrak{t}' \cdot 2$.

Example

Let $\mathsf{Ack} := (\mathcal{S}, \mathsf{V}, \to)$ be the TRS such that $\mathcal{S} := (\{z, s, k\}, \mathrm{ar})$ with $\mathrm{ar} \cdot z = 0$, $\mathrm{ar} \cdot s = 1$, and $\mathrm{ar} \cdot k = 2$, $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2\}$, $k z \mathsf{v}_1 \to s\mathsf{v}_1$, $k \lfloor s\mathsf{v}_1 \rfloor z \to k \mathsf{v}_1 \lfloor sz \rfloor$, and $k \lfloor s\mathsf{v}_1 \rfloor \lfloor s\mathsf{v}_2 \rfloor \to k \mathsf{v}_1 \lfloor k \lfloor s\mathsf{v}_1 \rfloor \mathsf{v}_2 \rfloor$.

Let $\succ$ be the binary relation on the underlying set of $\mathcal{S}$ satisfying $k \succ s$.

It can be easily checked that $k z \mathsf{v}_1 >_{\mathrm{LP}}^{\succ} s\mathsf{v}_1$, $k \lfloor s\mathsf{v}_1 \rfloor z >_{\mathrm{LP}}^{\succ} k \mathsf{v}_1 \lfloor sz \rfloor$, and $k \lfloor s\mathsf{v}_1 \rfloor \lfloor s\mathsf{v}_2 \rfloor >_{\mathrm{LP}}^{\succ} k \mathsf{v}_1 \lfloor k \lfloor s\mathsf{v}_1 \rfloor \mathsf{v}_2 \rfloor$. Therefore, Ack is terminating.

Exercise ●●○○○

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ such that $\mathcal{S} := (\{f, g, a\}, \mathrm{ar})$ with $\mathrm{ar}{\cdot}f = 1$, $\mathrm{ar}{\cdot}g = 1$, and $\mathrm{ar}{\cdot}a = 2$, $\mathsf{V} := \{\mathsf{v}\}$, and $f \underline{g\mathsf{v}} \to a \underline{g \underline{f\mathsf{v}}}\, \mathsf{v}$.

Show that there exists a binary relation $\succ$ on the underlying set of $\mathcal{S}$ such that $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.

Exercise ●●●○○

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \to)$ such that $\mathcal{S} := (\{f, g, h\}, \mathrm{ar})$ with $\mathrm{ar}{\cdot}f = 1$, $\mathrm{ar}{\cdot}g = 1$, and $\mathrm{ar}{\cdot}h = 1$, $\mathsf{V} := \{\mathsf{v}\}$, $g \underline{f\mathsf{v}} \to f \underline{h\mathsf{v}}$, and $hv \to gv$.

1. Show that the termination of $\mathcal{T}$ cannot be proven by the existence of a binary relation $\succ$ on the underlying set of $\mathcal{S}$ and an $\mathcal{S}, \mathsf{V}, \succ$-weight function ω so that $>_{\mathrm{KB}}^{\succ, \omega}$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.
2. Show that the termination of $\mathcal{T}$ cannot be proven by the existence of a binary relation $\succ$ on the underlying set of $\mathcal{S}$ so that $>_{\mathrm{LP}}^{\succ}$ is a simplification relation on $\mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathsf{V}$.
3. Show that the termination of $\mathcal{T}$ can be proven by using the polynomial interpretation method.

/

8. Confluence

8.1. Unification

Let $\mathcal{S}$ be a signature and V be a set of variables.

Given two $\mathcal{S}, \mathsf{V}$-substitutions σ and σ', σ is *more general* (or *less specific*) than σ' if there exists an $\mathcal{S}, \mathsf{V}$-substitution σ'' such that $\sigma' = \sigma'' \circ \sigma$. This property is denoted by $\sigma \leqslant_{\mathrm{g}} \sigma'$.

Examples

- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_1\mathsf{v}_3), (\mathsf{v}_2, \mathsf{c}_0)\}]$ and $\sigma' := [\{(\mathsf{v}_1, \mathsf{c}_1\underline{\mathsf{c}_2\mathsf{c}_0\mathsf{v}_1}), (\mathsf{v}_2, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1)\}]$. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ because $[\{(\mathsf{v}_3, \mathsf{c}_2\mathsf{c}_0\mathsf{v}_1)\}] \circ \sigma = \sigma'$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_0)\}]$ and $\sigma' := [\{(\mathsf{v}_2, \mathsf{c}_1\mathsf{v}_3)\}]$. We have $\sigma \not\leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \not\leqslant_{\mathrm{g}} \sigma$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitutions $\sigma := [\{(\mathsf{v}_1, \mathsf{c}_2\mathsf{v}_1\mathsf{v}_3)\}]$ and $\sigma' := [\{(\mathsf{v}_1, \mathsf{c}_2\mathsf{v}_3\mathsf{v}_1)\}]$. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \leqslant_{\mathrm{g}} \sigma$.

Proposition [Generality relation on $\mathcal{S}, \mathsf{V}$-substitutions]

Let $\mathcal{S}$ be a signature and V be a set of variables.

- The binary relation $\leqslant_{\mathrm{g}}$ is a preorder.
- Let σ and σ' be two $\mathcal{S}, \mathsf{V}$-substitutions. We have $\sigma \leqslant_{\mathrm{g}} \sigma'$ and $\sigma' \leqslant_{\mathrm{g}} \sigma$ iff there exists a renaming $\mathcal{S}, \mathsf{V}$-substitution σ'' such that $\sigma' = \sigma'' \circ \sigma$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms. If there exists an $\mathcal{S}, \mathsf{V}$-substitution σ such that $\overline{\sigma} \cdot \mathfrak{t} = \overline{\sigma} \cdot \mathfrak{t}'$, then

- the $\mathcal{S}, \mathsf{V}$-terms $\mathfrak{t}$ and $\mathfrak{t}'$ are *unifiable*. This property is denoted by $\mathfrak{t} \sim_{\mathrm{u}} \mathfrak{t}'$;
- the $\mathcal{S}, \mathsf{V}$-substitution σ is a *unifier* of $\mathfrak{t}$ and $\mathfrak{t}'$;
- when for any unifier σ' of $\mathfrak{t}$ and $\mathfrak{t}'$, $\sigma \leqslant_{\mathrm{g}} \sigma'$, σ is a *most general unifier* (*MGU*) of $\mathfrak{t}$ and $\mathfrak{t}'$;
- when σ is an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$, the $\mathcal{S}, \mathsf{V}$-term $\overline{\sigma} \cdot \mathfrak{t} = \overline{\sigma} \cdot \mathfrak{t}'$ is a *most general common instance* (*MGCI*) of $\mathfrak{t}$ and $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor$ and $\mathfrak{t}' := \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3$.

- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma_1 := [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor)\}]$.
  Since $\overline{\sigma_1} \cdot \mathfrak{t} = \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \rfloor = \overline{\sigma_1} \cdot \mathfrak{t}'$, σ_1 is a unifier of $\mathfrak{t}$ and $\mathfrak{t}'$.
  Moreover, σ_1 is an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$, and $\mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \rfloor$ is an MGCI of $\mathfrak{t}$ and $\mathfrak{t}'$.
- Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution $\sigma_2 := [\{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{c}_0), (\mathsf{v}_2, \mathsf{c}_0), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor)\}]$.
  Since $\overline{\sigma_2} \cdot \mathfrak{t} = \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \lfloor \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{c}_0 \rfloor \rfloor = \overline{\sigma_2} \cdot \mathfrak{t}'$, σ_2 is a unifier of $\mathfrak{t}$ and $\mathfrak{t}'$.
  Moreover, since $\sigma_2 = [\{(\mathsf{v}_2, \mathsf{c}_0)\}] \circ \sigma_1$, σ_2 is not an MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

An $\mathcal{S}, \mathsf{V}$-substitution σ is *idempotent* if $\sigma \circ \sigma = \sigma$.

Given an $\mathcal{S}, \mathsf{V}$-substitution σ, the *set of variables* of σ is the set

$$\mathrm{Vars} \cdot \sigma := \bigcup_{v \in \mathrm{Dom} \cdot \sigma} \mathrm{Vars} \cdot \lfloor \sigma \cdot v \rfloor.$$

Proposition [Idempotent $\mathcal{S}, \mathsf{V}$-substitutions]

Let $\mathcal{S}$ be a signature and V be a set of variables. An $\mathcal{S}, \mathsf{V}$-substitution σ is idempotent iff the sets $\mathrm{Dom} \cdot \sigma$ and $\mathrm{Vars} \cdot \sigma$ are disjoint.

Theorem [Idempotent MGUs]

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathsf{V}$-terms. If $\mathfrak{t}$ and $\mathfrak{t}'$ are unifiable, then there exists an idempotent MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

For any $S \in \mathcal{P} \cdot \llcorner \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \lrcorner^2$, let $\mathrm{Vars} \cdot S := \bigcup_{(\mathfrak{t}, \mathfrak{t}') \in S} \mathrm{Vars} \cdot \mathfrak{t} \cup \mathrm{Vars} \cdot \mathfrak{t}'$. Moreover, for any $S \in \mathcal{P} \cdot \llcorner \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \lrcorner^2$ and any $\mathcal{S}, \mathsf{V}$-substitution σ, let $\overline{\sigma} \cdot S := \{(\overline{\sigma} \cdot \mathfrak{t}, \overline{\sigma} \cdot \mathfrak{t}') : (\mathfrak{t}, \mathfrak{t}') \in S\}$.

Let the ARS $\mathsf{Unification}_{\mathcal{S}, \mathsf{V}} := \left(\{\mathsf{Fail}\} \cup \mathcal{P} \cdot \llcorner \mathfrak{T} \cdot \mathcal{S} \cdot \mathsf{V} \lrcorner^2, \Rightarrow\right)$ such that

1. [Simplification] $\{(v, v)\} \sqcup S \Rightarrow S$ if $v \in \mathsf{V}$;
2. [Decomposition] $\{(c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, c\, \mathfrak{t}'_1 \dots \mathfrak{t}'_n)\} \sqcup S \Rightarrow \{(\mathfrak{t}_1, \mathfrak{t}'_1), \dots, (\mathfrak{t}_n, \mathfrak{t}'_n)\} \cup S$;
3. [Orientation] $\{(\mathfrak{t}, v)\} \sqcup S \Rightarrow \{(v, \mathfrak{t})\} \cup S$ if $\mathfrak{t} \notin \mathsf{V}$ and $v \in \mathsf{V}$;
4. [Variable elimination] $\{(v, \mathfrak{t})\} \sqcup S \Rightarrow \{(v, \mathfrak{t})\} \cup \overline{[\{(v, \mathfrak{t})\}]} \cdot S$ if $v \notin \mathrm{Vars} \cdot \mathfrak{t}$ and $v \in \mathrm{Vars} \cdot S$;
5. [Recursive variable occurrence] $\{(v, \mathfrak{t})\} \sqcup S \Rightarrow \mathsf{Fail}$ if $\mathfrak{t} \notin \mathsf{V}$ and $v \in \mathrm{Vars} \cdot \mathfrak{t}$;
6. [Constant clash] $\{(c\, \mathfrak{t}_1 \dots \mathfrak{t}_n, c'\, \mathfrak{t}'_1 \dots \mathfrak{t}'_{n'})\} \sqcup S \Rightarrow \mathsf{Fail}$ if $c \neq c'$.

This ARS leads to the Martelli-Montanari unification Algorithm [A. Martelli, U. Montanari, Unification in linear time and space: a structured presentation, 1976].

Let $\mathcal{S}$ be a signature, V be a set of variables, and t and t' be two $\mathcal{S}, \mathsf{V}$-terms.

The ARS $\mathsf{Unification}_{\mathcal{S}, \mathsf{V}}$ is used by computing a **normal form** of $\{(\mathsf{t}, \mathsf{t}')\}$ and, when this normal form is a set S, by considering the $\mathcal{S}, \mathsf{V}$-**substitution** $[S]$ **specified** by S.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathsf{t} := \mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor$ and $\mathsf{t}' := \mathsf{c}_2 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3$. In $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}}$,

$$\{(\mathsf{t}, \mathsf{t}')\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1, \mathsf{v}_3)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{v}_3, \mathsf{c}_2 \mathsf{c}_0 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor)\} =: \sigma.$$

We can check that the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-substitution σ is an idempotent MGU of t and t'.

Example

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathsf{t} := \mathsf{c}_3 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{v}_2 \mathsf{c}_0 \rfloor \mathsf{v}_1$ and $\mathsf{t}' := \mathsf{c}_3 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \mathsf{v}_2$. In $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}}$,

$$\{(\mathsf{t}, \mathsf{t}')\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{c}_2 \mathsf{v}_2 \mathsf{c}_0, \mathsf{v}_3), (\mathsf{v}_1, \mathsf{v}_2)\} \Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{c}_2 \mathsf{v}_2 \mathsf{c}_0, \mathsf{v}_3), (\mathsf{c}_1 \mathsf{v}_2, \mathsf{v}_2)\}$$
$$\Rightarrow \{(\mathsf{v}_1, \mathsf{c}_1 \mathsf{v}_2), (\mathsf{c}_2 \mathsf{v}_2 \mathsf{c}_0, \mathsf{v}_3), (\mathsf{v}_2, \mathsf{c}_1 \mathsf{v}_2)\} \Rightarrow \mathsf{Fail}.$$

We can check that t and t' are not unifiable.

Due to the following results, the ARS $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$, where $\mathcal{S}$ is any signature and V is any set of variables, computes exactly what it is designed to compute.

Theorem [Termination of $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$]

For any signature $\mathcal{S}$ and set of variables V, the ARS $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$ is terminating.

Exercise ●●○○○○

Show that $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$ is not convergent, where $\mathcal{S}$ is a signature and V is a set of variables.

Proposition [Computation of $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$]

Let $\mathcal{S}$ be a signature, V be a set of variables, $\mathfrak{t}, \mathfrak{t}' \in \mathfrak{T}{\cdot}\mathcal{S}{\cdot}\mathrm{V}$, and $S := \{(\mathfrak{t}, \mathfrak{t}')\}$.

- If $\mathfrak{t} \sim_{\mathrm{u}} \mathfrak{t}'$, then any normal form of S in $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$ is a set S' such that $[S']$ is an idempotent MGU of $\mathfrak{t}$ and $\mathfrak{t}'$.
- Otherwise, the unique normal form of S in $\mathsf{Unification}_{\mathcal{S},\mathrm{V}}$ is Fail.

Exercise ●○○○○

By using the ARS $\mathsf{Unification}_{\mathcal{S}_{\mathbb{N}^2}, \mathrm{V}_{\mathbb{N}}}$, compute a normal form of $\{(\mathfrak{t}, \mathfrak{t}')\}$ where

1. $\mathfrak{t} := \mathsf{c}_3 \mathsf{v}_1 \lfloor \mathsf{c}_2 \mathsf{c}_0 \mathsf{v}_1 \rfloor \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor$ and $\mathfrak{t}' := \mathsf{c}_3 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor \mathsf{v}_3 \mathsf{v}_4$;
2. $\mathfrak{t} := \mathsf{c}_2 \mathsf{v}_1 \lfloor \mathsf{c}_1 \mathsf{v}_2 \rfloor$ and $\mathfrak{t}' := \mathsf{c}_2 \lfloor \mathsf{c}_2 \mathsf{v}_3 \mathsf{v}_2 \rfloor \mathsf{v}_1$.

Exercise ●●○○○

Describe the relation between the ARSs $\mathsf{Matching}_{\mathcal{S}, \mathrm{V}}$ and $\mathsf{Unification}_{\mathcal{S}, \mathrm{V}}$ where $\mathcal{S}$ is a signature and V is a set of variables.

Exercise ●●●○○

Let $\mathcal{S}$ be a signature and V be a set of variables. Let $\mathfrak{t}$ and $\mathfrak{t}'$ be two $\mathcal{S}, \mathrm{V}$-terms such that $\mathfrak{t} \sim_{\mathrm{u}} \mathfrak{t}'$. Show that an MGCI of $\mathfrak{t}$ and $\mathfrak{t}'$ is a supremum of $\mathfrak{t}$ and $\mathfrak{t}'$ for the preorder $\preccurlyeq_{\mathrm{p}}$.

8.2. Overlaps and critical data

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{r}$ be a $\mathcal{S}, \mathsf{V}$-term admitting the two decompositions

$$\triangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \sigma_1 = \mathfrak{r} = \triangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}_2 \cdot \sigma_2$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are holed $\mathcal{S}, \mathsf{V}$-terms, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are $\mathcal{S}, \mathsf{V}$-terms, and σ_1 and σ_2 are $\mathcal{S}, \mathsf{V}$-substitutions. Let w_1 (resp. w_2) be the hole position of $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$). Without loss of generality, let us consider that $\ell \cdot w_1 \leqslant \ell \cdot w_2$.

We distinguish the following disjoint and exhaustive three cases:

1. [Horizontal disjunction] This occurs when w_1 is not a prefix of w_2;
2. [Vertical disjunction] This occurs when w_1 is a prefix of w_2, and, by setting u as the word over positive integers such that $w_2 = w_1 \centerdot u$, u is not the position of an internal node of $\mathfrak{t}_1$;
3. [Overlap] This occurs when w_1 is a prefix of w_2, and, by setting u as the word over positive integers such that $w_2 = w_1 \centerdot u$, u is the position of an internal node of $\mathfrak{t}_1$.

Note that when $w_1 = w_2$, we are in the Vertical disjunction Case if $\mathfrak{t}_1$ is a variable and in the Overlap Case otherwise.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS.

Let $\mathfrak{r}$ be an $\mathcal{S}, \mathsf{V}$-term, and w_1 and w_2 be two positions within $\mathfrak{r}$ such that $\ell \cdot w_1 \leqslant \ell \cdot w_2$, $\mathfrak{r} \Rightarrow_{w_1} \mathfrak{u}_1$, and $\mathfrak{r} \Rightarrow_{w_2} \mathfrak{u}_2$ where $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are two $\mathcal{S}, \mathsf{V}$-terms. By Proposition [Rewrite relation of a TRS],

$$\mathfrak{r} = \triangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \sigma_1 \quad \Rightarrow_{w_1} \quad \triangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}'_1 \cdot \sigma_1 = \mathfrak{u}_1$$

and

$$\mathfrak{r} = \triangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}_2 \cdot \sigma_2 \quad \Rightarrow_{w_2} \quad \triangle \cdot \mathfrak{s}_2 \cdot \mathfrak{t}'_2 \cdot \sigma_2 = \mathfrak{u}_2$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are holed $\mathcal{S}, \mathsf{V}$-terms, $\mathfrak{t}_1$, $\mathfrak{t}'_1$, $\mathfrak{t}_2$, and $\mathfrak{t}'_2$ are $\mathcal{S}, \mathsf{V}$-terms such that $\mathfrak{t}_1 \rightarrow \mathfrak{t}'_1$ and $\mathfrak{t}_2 \rightarrow \mathfrak{t}'_2$, and σ_1 and σ_2 are $\mathcal{S}, \mathsf{V}$-substitutions.

Let us understand what happens in each of the three previous cases.

Let us consider the previous notations and definitions.

In the **Horizontal disjunction Case**, w_1 is not a prefix of w_2.

Therefore, by considering that $\square_1$ and $\square_2$ are two variables which do not belong to V, we have

$$\mathfrak{r} = \mathfrak{s}[\{(\square_1, \overline{\sigma_1} \cdot \mathfrak{t}_1), (\square_2, \overline{\sigma_2} \cdot \mathfrak{t}_2)\}]$$

where $\mathfrak{s}$ is an $\mathcal{S}, \mathsf{V} \sqcup \{\square_1, \square_2\}$-term having exactly one occurrence of $\square_1$ and exactly one occurrence of $\square_2$ at respective positions w_1 and w_2.

We have

$$\mathfrak{u}_1 \Rightarrow_{w_2} \mathfrak{u}$$

and

$$\mathfrak{u}_2 \Rightarrow_{w_1} \mathfrak{u}$$

where $\mathfrak{u}$ is the $\mathcal{S}, \mathsf{V}$-term defined by

$$\mathfrak{u} := \mathfrak{s}[\{(\square_1, \overline{\sigma_1} \cdot \mathfrak{t}'_1), (\square_2, \overline{\sigma_2} \cdot \mathfrak{t}'_2)\}].$$

Let us consider the previous notations and definitions.

In the Vertical disjunction Case, w_1 is a prefix of w_2, and, by writing $w_2 = w_1 \centerdot u$, u is not the position of an internal node of $\mathfrak{t}_1$.

There exist $v \in \mathsf{V}$ and two words u_0 and p such that $u = u_0 \centerdot p$, $u_0 \in \mathrm{P}\cdot\mathfrak{t}_1$, and $\mathfrak{t}_1 \cdot u_0 = v$.

Moreover, there exist a holed $\mathcal{S}, \mathsf{V}$-term $\mathfrak{q}$ and an $\mathcal{S}, \mathsf{V}$-substitution ρ such that

$$\sigma_1 \cdot v = \triangle \cdot \mathfrak{q} \cdot \mathfrak{t}_2 \cdot \rho \quad \Rightarrow_p \quad \triangle \cdot \mathfrak{q} \cdot \mathfrak{t}_2' \cdot \rho.$$

Let τ be the $\mathcal{S}, \mathsf{V}$-substitution defined by $\tau \cdot v := \triangle \cdot \mathfrak{q} \cdot \mathfrak{t}_2' \cdot \rho$, and $\tau \cdot v' := \sigma_1 \cdot v'$ for any $v' \in \mathsf{V} \setminus \{v\}$.

Let $n := \ell_v \cdot \mathfrak{t}_1$ (resp. $n' := \ell_v \cdot \mathfrak{t}_1'$) and $\{u_0, \dots, u_{n-1}\}$ (resp. $\{u_0', \dots, u_{n'-1}'\}$) be the set of positions of the variable v in $\mathfrak{t}_1$ (resp. $\mathfrak{t}_1'$). We have

$$\mathfrak{u}_1 \quad \Rightarrow_{w_1 \centerdot u_0' \centerdot p} \quad \cdots \quad \Rightarrow_{w_1 \centerdot u_{n'-1}' \centerdot p} \quad \mathfrak{u}$$

and

$$\mathfrak{u}_2 \quad \Rightarrow_{w_1 \centerdot u_1 \centerdot p} \quad \cdots \quad \Rightarrow_{w_1 \centerdot u_{n-1} \centerdot p} \quad \triangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1 \cdot \tau \quad \Rightarrow_{w_1} \quad \mathfrak{u}$$

where $\mathfrak{u}$ is the $\mathcal{S}, \mathsf{V}$-term defined by

$$\mathfrak{u} := \triangle \cdot \mathfrak{s}_1 \cdot \mathfrak{t}_1' \cdot \tau.$$

Let us consider the previous notations and definitions.

In the Overlap Case, w_1 is a prefix of w_2, and, by writing $w_2 = w_1 . u$, u is the position of an internal node of $\mathfrak{t}_1$.

In this case, there is no generic way to exhibit an $\mathcal{S}, \mathsf{V}$-term $\mathfrak{u}$ such that $\mathfrak{u}_1 \Rightarrow^* \mathfrak{u}$ and $\mathfrak{u}_2 \Rightarrow^* \mathfrak{u}$.

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS.

We shall describe a necessary and sufficient condition for the fact that $\mathcal{T}$ is locally confluent.

When $\mathcal{T}$ is terminating and $\rightarrow$ is finite, this condition leads to an algorithm to decide local confluence of $\mathcal{T}$.

Let $\mathcal{S}$ be a signature and V be a set of variables.

Let $\mathfrak{t}$ and $\mathfrak{t}'$ be $\mathcal{S}, \mathsf{V}$-terms such that $\mathrm{Vars} \cdot \mathfrak{t} \cap \mathrm{Vars} \cdot \mathfrak{t}' = \emptyset$. If there exists a position u within $\mathfrak{t}'$ such that $\mathfrak{t}' \cdot u$ is not a leaf, and $\mathfrak{t}' \cdot u$ and $\mathfrak{t}$ are unifiable, then

- $\mathfrak{t}$ *overlaps* $\mathfrak{t}'$;
- u is the *overlapping* position of $\mathfrak{t}$ in $\mathfrak{t}'$;
- if σ is an MGU of $\mathfrak{t}' \cdot u$ and $\mathfrak{t}$, then $\overline{\sigma} \cdot \mathfrak{t}'$ is the *fusion* of $\mathfrak{t}$ at position u into $\mathfrak{t}'$.

Examples

Let the $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-terms $\mathfrak{t}' := \mathsf{c}_{2,0} \underline{\mathsf{c}_{2,1} \mathsf{v}_1 \underline{\mathsf{c}_{2,1} \mathsf{v}_2 \mathsf{v}_3}} \mathsf{v}_4$ and $\mathfrak{t} := \mathsf{c}_{2,1} \underline{\mathsf{c}_{2,0} \mathsf{v}_5 \mathsf{v}_6} \underline{\mathsf{c}_{2,1} \mathsf{v}_7 \mathsf{v}_8}$.

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ overlaps $\mathfrak{t}'$ at position $u := 1$.

  The fusion of $\mathfrak{t}$ at position u into $\mathfrak{t}'$ is $\mathsf{c}_{2,0} \underline{\mathsf{c}_{2,1} \underline{\mathsf{c}_{2,0} \mathsf{v}_5 \mathsf{v}_6} \underline{\mathsf{c}_{2,1} \mathsf{v}_7 \mathsf{v}_8}} \mathsf{v}_4$.

- The $\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}$-term $\mathfrak{t}$ overlaps $\mathfrak{t}'$ at position $u := 12$.

  The fusion of $\mathfrak{t}$ at position u into $\mathfrak{t}'$ is $\mathsf{c}_{2,0} \underline{\mathsf{c}_{2,1} \mathsf{v}_1 \underline{\mathsf{c}_{2,1} \underline{\mathsf{c}_{2,0} \mathsf{v}_5 \mathsf{v}_6} \underline{\mathsf{c}_{2,1} \mathsf{v}_7 \mathsf{v}_8}}} \mathsf{v}_4$.