For any set of variables V and any $k \in \mathbb{N}$, set $\mathsf{V}^{(k)}$ as the set of variables $\{v^{(k)} : v \in \mathsf{V}\}$. We identify $\mathsf{V}^{(0)}$ with V. When $k \geqslant 1$, $\mathsf{V}^{(k)}$ is a distinct copy of V. Let also $\mathrm{C}\cdot\mathsf{V} := \bigsqcup_{k \in \mathbb{N}} \mathsf{V}^{(k)}$.

Example

Let the set of variables $\mathsf{V} := \{\mathsf{v}_1, \mathsf{v}_2\}$. We have $\mathrm{C}\cdot\mathsf{V} = \left\{\mathsf{v}_1^{(0)} = \mathsf{v}_1, \mathsf{v}_2^{(0)} = \mathsf{v}_2, \mathsf{v}_1^{(1)}, \mathsf{v}_2^{(1)}, \mathsf{v}_1^{(2)}, \mathsf{v}_2^{(2)}, \mathsf{v}_1^{(3)}, \mathsf{v}_2^{(3)}, \dots\right\}$.

Let $\mathcal{S}$ be a signature. For any $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-term $\mathfrak{t}$, the *shift* of $\mathfrak{t}$ is the $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-term $\mathrm{shift}\cdot\mathfrak{t} := \overline{\sigma}\cdot\mathfrak{t}$ where σ is any $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-substitution satisfying $\sigma\cdot v^{(k)} = v^{(1+k)}$ for any $v^{(k)} \in \mathrm{C}\cdot\mathsf{V}$.

Example

By considering the previous set of variables V, $\mathrm{shift}\cdot\mathsf{a}_2 \lfloor \mathsf{a}_1 \mathsf{v}_1 \rfloor \lfloor \mathsf{a}_2 \mathsf{v}_1^{(3)} \mathsf{v}_2^{(1)} \rfloor = \mathsf{a}_2 \lfloor \mathsf{a}_1 \mathsf{v}_1^{(1)} \rfloor \lfloor \mathsf{a}_2 \mathsf{v}_1^{(4)} \mathsf{v}_2^{(2)} \rfloor$.

The *disjunction* of a pair $(\mathfrak{t}, \mathfrak{t}')$ of $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-terms is the pair $\left(\mathfrak{t}, \mathfrak{t}'^{\bullet}\right)$ where $\mathfrak{t}'^{\bullet} := \mathrm{shift}^{\ell}\cdot\mathfrak{t}'$ and ℓ is the smallest natural number such that all superscripts of the variables of $\mathfrak{t}$ are smaller than all superscripts of the variables of $\mathfrak{t}'^{\bullet}$.

Example

The disjunction of the pair $\left(\mathsf{a}_2 \mathsf{v}_1 \mathsf{v}_2^{(3)}, \mathsf{a}_3 \mathsf{v}_1^{(1)} \mathsf{v}_2 \mathsf{v}_1^{(5)}\right)$ is $\left(\mathsf{a}_2 \mathsf{v}_1 \mathsf{v}_2^{(3)}, \mathsf{a}_3 \mathsf{v}_1^{(5)} \mathsf{v}_2^{(4)} \mathsf{v}_1^{(9)}\right)$.

For any TRS $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$, let the TRS $\mathrm{C}\cdot\mathcal{T} := (\mathcal{S}, \mathrm{C}\cdot\mathrm{V}, \to)$ where this second occurrence of $\to$ is the elementary rewrite relation of $\mathcal{T}$ extended to the set $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathrm{V}\rfloor$. In this way, the rewrite relation $\Rightarrow$ of $\mathrm{C}\cdot\mathcal{T}$ is the rewrite relation of $\mathcal{T}$ extended to $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor\mathrm{C}\cdot\mathrm{V}\rfloor$.

Definition

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. Let $r_1 := (\mathfrak{t}_1, \mathfrak{t}'_1)$ and $r_2 := (\mathfrak{t}_2, \mathfrak{t}'_2)$ be two rewrite rules of $\mathrm{C}\cdot\mathcal{T}$ such that $\mathfrak{t}_2^\bullet$ overlaps $\mathfrak{t}_1$ at an overlapping position u, where $(\mathfrak{t}_1, \mathfrak{t}_2^\bullet)$ is the disjunction of $(\mathfrak{t}_1, \mathfrak{t}_2)$.

- A *critical data* of $\mathcal{T}$ is a triple (r_1, u, r_2) such that $r_1 \neq r_2$ or $u \neq \epsilon$.
- The *critical term* associated with the critical data (r_1, u, r_2) of $\mathcal{T}$ is the fusion of $\mathfrak{t}_2^\bullet$ at position u into $\mathfrak{t}_1$. This critical term is an $\mathcal{S}, \mathrm{C}\cdot\mathrm{V}$-term.
- The *critical pair* associated with the critical data (r_1, u, r_2) of $\mathcal{T}$ is the pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of $\mathcal{S}, \mathrm{C}\cdot\mathrm{V}$-terms such that, by denoting by $\mathfrak{t}$ the critical term associated with (r_1, u, r_2), $\mathfrak{s}_1$ is obtained by a one-step rewrite at root from $\mathfrak{t}$ by using r_1 in $\mathrm{C}\cdot\mathcal{T}$, and $\mathfrak{s}_2$ is obtained by a one-step rewrite at position u from $\mathfrak{t}$ by using r_2 in $\mathrm{C}\cdot\mathcal{T}$.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}_{\mathbb{N}^2}, \mathsf{V}_{\mathbb{N}}, \to)$ such that

$$r_1 := \mathsf{c}_{2,0} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_1 \underbracket{\mathsf{c}_{2,1} \mathsf{v}_2 \mathsf{v}_3}} \mathsf{v}_4 \to \mathsf{c}_3 \mathsf{v}_1 \mathsf{v}_2 \mathsf{v}_3$$

and

$$r_2 := \mathsf{c}_{2,1} \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1 \mathsf{v}_2} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_3 \mathsf{v}_4} \to \mathsf{c}_1 \mathsf{v}_1.$$

- ☐ The triple $(r_1, 1, r_2)$ is a critical data of $\mathcal{T}$.
- ☐ The $\mathcal{S}_{\mathbb{N}^2}, \mathrm{C} \cdot \mathsf{V}_{\mathbb{N}}$-term

$$\mathfrak{t} := \mathsf{c}_{2,0} \underbracket{\mathsf{c}_{2,1} \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1^{(1)} \mathsf{v}_2^{(1)}} \underbracket{\mathsf{c}_{2,1} \mathsf{v}_3^{(1)} \mathsf{v}_4^{(1)}}} \mathsf{v}_4$$

  is the critical term associated with this critical data.

- ☐ The critical pair associated with this critical data is

$$\left(\mathsf{c}_3 \underbracket{\mathsf{c}_{2,0} \mathsf{v}_1^{(1)} \mathsf{v}_2^{(1)}} \mathsf{v}_3^{(1)} \mathsf{v}_4^{(1)}, \mathsf{c}_{2,0} \underbracket{\mathsf{c}_1 \mathsf{v}_1^{(1)}} \mathsf{v}_4 \right).$$

A critical data (r_1, u, r_2) of a TRS $\mathcal{T}$ is *joinable* if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are joinable in $\mathrm{C}{\cdot}\mathcal{T}$, where $(\mathfrak{s}_1, \mathfrak{s}_2)$ is the critical pair associated with (r_1, u, r_2).

Theorem [Critical pairs and local confluence]

A TRS $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \rightarrow)$ is locally confluent iff all critical data of $\mathcal{T}$ are joinable.

This leads to the following algorithm to prove that a TRS $\mathcal{T}$ having a finite elementary rewrite relation is confluent:

(1) Prove that $\mathcal{T}$ is terminating (for instance, by one of the methods seen in the previous chapter).

(2) List all critical data of $\mathcal{T}$.

(3) For each critical pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of these critical data, compute the future of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathrm{C}{\cdot}\mathcal{T}$ and exhibit a common element.

By Theorems [Newman's Lemma] and [Critical pairs and local confluence], these steps show the confluence of $\mathcal{T}$.

The fact that the elementary rewrite relation of $\mathcal{T}$ is finite implies that $\mathcal{T}$ admits finitely many critical data. Moreover, Step (1) implies that the computation of the future of any elements of a critical pair ends. Therefore, the previous steps form an algorithm.

Example

Let $\mathsf{DA} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing two binary constants a and b, and $\to$ is defined through the four rewrite rules

- $r_1 := a\lfloor a12\rfloor 3 \to a1\lfloor a23\rfloor$;
- $r_2 := b\lfloor a12\rfloor 3 \to a1\lfloor b23\rfloor$;
- $r_3 := b1\lfloor a23\rfloor \to b\lfloor b12\rfloor 3$;
- $r_4 := b1\lfloor b2\lfloor b34\rfloor\rfloor \to a\lfloor b1\lfloor b23\rfloor\rfloor 4$.

This TRS is introduced in [S. Giraudo, Combinatorial operads from monoids, 2015].

We assume that we have proven that DA is terminating.

This TRS contains the following critical data: $(r_1, 1, r_1)$, $(r_2, 1, r_1)$, (r_2, ϵ, r_3), (r_2, ϵ, r_4), $(r_3, 2, r_1)$, (r_3, ϵ, r_2), (r_4, ϵ, r_2), $(r_4, 2, r_2)$, $(r_4, 22, r_2)$, $(r_4, 22, r_3)$, $(r_4, 2, r_4)$, and $(r_4, 22, r_4)$.

The critical data $(r_4, 2, r_4)$ has $b1\lfloor b1^{(1)}\lfloor b2^{(1)}\lfloor b3^{(1)}4^{(1)}\rfloor\rfloor\rfloor$ as critical term and is joinable since

$$b1\lfloor b1^{(1)}\lfloor b2^{(1)}\lfloor b3^{(1)}4^{(1)}\rfloor\rfloor\rfloor \Rightarrow a\lfloor b1\lfloor b1^{(1)}2^{(1)}\rfloor\rfloor\lfloor b3^{(1)}4^{(1)}\rfloor =: \mathfrak{s}$$

and

$$b1\lfloor b1^{(1)}\lfloor b2^{(1)}\lfloor b3^{(1)}4^{(1)}\rfloor\rfloor\rfloor \Rightarrow b1\lfloor a\lfloor b1^{(1)}\lfloor b2^{(1)}3^{(1)}\rfloor\rfloor 4^{(1)}\rfloor \Rightarrow b\lfloor b1\lfloor b1^{(1)}\lfloor b2^{(1)}3^{(1)}\rfloor\rfloor 4^{(1)}\rfloor \Rightarrow b\lfloor a\lfloor b1\lfloor b1^{(1)}2^{(1)}\rfloor\rfloor 3^{(1)}\rfloor 4^{(1)} \Rightarrow \mathfrak{s}.$$

Exercise ●●○○○

Show that all other critical data of DA are joinable.

Exercise ●○○○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing two unary constants f and g, and $\to$ is defined through the two rewrite rules $f \lfloor g \lfloor f1 \rfloor \rfloor \to 1$ and $f \lfloor g1 \rfloor \to g \lfloor f1 \rfloor$. List all critical data of $\mathcal{T}$.

Exercise ●●○○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant u, a unary constant i, and a binary constant m, and $\to$ is defined through the three rewrite rules $mu1 \to 1$, $m \lfloor i1 \rfloor 1 \to u$, $m \lfloor m12 \rfloor 3 \to m1 \lfloor m23 \rfloor$.

1. List all critical data of $\mathcal{T}$.
2. Show that there are some critical data of $\mathcal{T}$ which are not joinable.

Exercise ●●●○○

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant e and two binary constants m and s, and $\to$ is defined through the seven rewrite rules $m1e \to 1$, $me1 \to 1$, $s1e \to 1$, $se1 \to e$, $s11 \to e$, $s \lfloor m12 \rfloor 3 \to m \lfloor s13 \rfloor \lfloor s2 \lfloor s31 \rfloor \rfloor$, and $s1 \lfloor m23 \rfloor \to s \lfloor s12 \rfloor 3$.

Apply the previously described algorithm to show that $\mathcal{T}$ is confluent.

8.3. Orthogonality

A critical data (r_1, u, r_2) of a TRS $\mathcal{T}$ is *trivial* if $\mathfrak{s}_1 = \mathfrak{s}_2$ where $(\mathfrak{s}_1, \mathfrak{s}_2)$ is the critical pair associated with (r_1, u, r_2).

Definition

Let $\mathcal{T} := (\mathcal{S}, \mathrm{V}, \to)$ be a TRS. When

- all left-hand sides of rewrite rules of $\mathcal{T}$ are linear, $\mathcal{T}$ is *left-linear*;
- $\mathcal{T}$ admits no critical data, $\mathcal{T}$ is *non-overlapping*;
- all critical data of $\mathcal{T}$ are trivial, $\mathcal{T}$ is *weakly non-overlapping*;
- $\mathcal{T}$ is (weakly) non-overlapping and left-linear, $\mathcal{T}$ is *(weakly) orthogonal*.

Examples

Let $\mathcal{S}$ be the signature containing two nullary constants t and f, and one binary constant o.

- Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\to$ is defined through the three rewrite rules $r_1 := ot1 \to t$, $r_2 := o1t \to t$, and $off \to f$. The critical data (r_1, ϵ, r_2) is trivial and $\mathcal{T}$ is weakly orthogonal.
- Let $\mathcal{T}' := (\mathcal{S}, \mathbb{N}, \to')$ be the TRS such that $\to'$ is defined through the two rewrite rules $ot1 \to' t$ and $of1 \to' 1$. The TRS $\mathcal{T}'$ is orthogonal.

Theorem [Confluence of weakly orthogonal TRSs

Any weakly orthogonal TRS is confluent.

This result is important to establish the confluence of TRSs which are not terminating.

The condition to be left-linear for a weakly orthogonal TRS is necessary.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \rightarrow)$ be the TRS such that $\mathcal{S}$ is the signature containing three nullary constants i, t, and f, a unary constant s, and a binary constant e, and $\rightarrow$ is defined through the three rewrite rules $i \rightarrow si$, $e11 \rightarrow t$, and $e1\lfloor s1 \rfloor \rightarrow f$.

This TRS $\mathcal{T}$ is non-overlapping but is not left-linear.

Since

$$eii \Rightarrow t \quad \text{and} \quad eii \Rightarrow ei\lfloor si \rfloor \Rightarrow f,$$

the $\mathcal{S}, \mathbb{N}$-term eii has two normal forms. Therefore, $\mathcal{T}$ is not confluent.

Example

Let $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ be the TRS such that $\mathcal{S}$ is the signature containing a nullary constant a, a unary constant k, and binary constant f, and $\to$ is defined through the three rewrite rules $r_1 := fa1 \to k1$, $r_2 := f1a \to k1$, and $k1 \to fa1$.

First, $\mathcal{T}$ is not terminating because

$$k1 \Rightarrow fa1 \Rightarrow k1 \Rightarrow fa1 \Rightarrow \cdots$$

leads to an infinite rewrite sequence in $\mathcal{T}$.

Second, all critical data of $\mathcal{T}$ are trivial. Indeed, the only critical data of $\mathcal{T}$ are (r_1, ϵ, r_2) and (r_2, ϵ, r_1). For both these critical data, the associated critical term is faa, and the associated critical pair is (ka, ka).

Therefore, by Theorem [Confluence of weakly orthogonal TRSs], $\mathcal{T}$ is confluent.

8.4. Completion

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$.

Let the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow} := \left(\{\mathsf{Fail}\} \cup \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor\rfloor^2, \Rightarrow\right)$ such that, for any $S \in \mathcal{P}\cdot\lfloor\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor\rfloor^2$,

1. [Critical pair join]

$$S \Rightarrow S \sqcup \{(\mathfrak{t}_1, \mathfrak{t}_2)\}$$

   if there exists a critical pair $(\mathfrak{s}_1, \mathfrak{s}_2)$ of a critical data of the TRS $\mathcal{T} := (\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$, such that $\mathfrak{t}_1$ is a normal form of $\mathfrak{s}_1$ in $\mathcal{T}$ and $\mathfrak{t}_2$ is a normal form of $\mathfrak{s}_2$ in $\mathcal{T}$, and $\mathfrak{t}_1 \neq \mathfrak{t}_2$.

2. [Reduction clash]

$$S \Rightarrow \mathsf{Fail}$$

   if there exists $(\mathfrak{t}_1, \mathfrak{t}_2) \in S$ such that $\mathfrak{t}_1 \not\rightsquigarrow \mathfrak{t}_2$.

This ARS leads to the Knuth-Bendix completion Algorithm [D. Knuth, P. Bendix, Simple Words Problems in Universal Algebras, 1970].

The original Knuth-Bendix completion Algorithm works on sets of identities (unordered pairs of $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-terms) rather than elementary rewrite relations (ordered pairs of $\mathcal{S}, \mathrm{C}\cdot\mathsf{V}$-terms).

Let $\mathcal{S}$ be a signature, V be a set of variables, and $\rightsquigarrow$ be a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\llcorner \mathrm{C}\cdot\mathsf{V}\lrcorner$.

Given a TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, S)$, the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is used by computing a normal form of the set S and, when this normal form is a set S', by considering the TRS $(\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S')$ as the result.

Example

Let $\mathcal{S}$ be the signature containing two binary constants a and b. Let us consider the set of variables $\mathbb{N}$. Let $\rightsquigarrow$ be the $\succ, \omega$-Knuth-Bendix relation where $\succ$ satisfies $b \succ a$, and ω is the $\mathcal{S}, \mathrm{C}\cdot\mathbb{N}, \succ$-weight function defined by $\omega_{\mathrm{C}\cdot\mathbb{N}} := 1$, $\omega\cdot a = 3$, and $\omega\cdot b = 2$.

Let $S := \{(a\llcorner a12\lrcorner 3, a1\llcorner b23\lrcorner)\}$. In $\mathsf{Completion}_{\mathcal{S},\mathbb{N},\rightsquigarrow}$, we have

$$S \Rightarrow \{(a\llcorner a12\lrcorner 3, a1\llcorner b23\lrcorner), (\mathfrak{t}_1, \mathfrak{t}_2)\}$$

with the following definitions.

By denoting by r the element of S, we have that $(r, 1, r)$ is a critical data of the TRS $(\mathcal{S}, \mathsf{V}, S)$. The critical term associated with this critical data is $\mathfrak{t} := a\llcorner a\llcorner a1^{(1)}2^{(1)}\lrcorner 3^{(1)}\lrcorner 3$. The associated critical pair is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := a\llcorner a1^{(1)}2^{(1)}\lrcorner\llcorner b3^{(1)}3\lrcorner$ and $\mathfrak{s}_2 := a\llcorner a1^{(1)}\llcorner b2^{(1)}3^{(1)}\lrcorner\lrcorner 3$.

We have moreover

$$\mathfrak{s}_1 \Rightarrow a1^{(1)}\llcorner b2^{(1)}\llcorner b3^{(1)}3\lrcorner\lrcorner =: \mathfrak{t}_2 \quad \text{and} \quad \mathfrak{s}_2 \Rightarrow a1^{(1)}\llcorner b\llcorner b2^{(1)}3^{(1)}\lrcorner 3\lrcorner =: \mathfrak{t}_1.$$

These two normal forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ satisfy $\mathfrak{t}_1 \rightsquigarrow \mathfrak{t}_2$.

Due to the following results, the ARS $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$, where $\mathcal{S}$ is a signature, V is a set of variables, and $\rightsquigarrow$ is a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$, computes exactly what it is designed to compute.

Exercise ●●●○○

Show that $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is neither terminating nor confluent, where $\mathcal{S}$ is a signature, V is a set of variables, and $\rightsquigarrow$ is a reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$.

Theorem [Normal forms of $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$]

Let $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$ be a TRS and $\rightsquigarrow$ be a $\mathcal{T}$-compatible reduction relation on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$. All normal forms of the set $\rightarrow$ in $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ which are different of Fail are binary relations S on $\mathfrak{T}\cdot\mathcal{S}\cdot\lfloor \mathrm{C}\cdot\mathsf{V}\rfloor$ such that $\mathcal{T}' := (\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ is a convergent TRS such that $\equiv_{\mathcal{T}'} = \equiv_{\mathrm{C}\cdot\mathcal{T}}$.

In other terms, given a terminating TRS $\mathcal{T} := (\mathcal{S}, \mathsf{V}, \rightarrow)$, a normal form of of $\rightarrow$ in $\mathsf{Completion}_{\mathcal{S},\mathsf{V},\rightsquigarrow}$ is an elementary rewrite relation S such that the TRSs $(\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ and $\mathrm{C}\cdot\mathcal{T}$ are convertibility equivalent, and $(\mathcal{S}, \mathrm{C}\cdot\mathsf{V}, S)$ is convergent (even if $\mathrm{C}\cdot\mathcal{T}$ is not).

Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m, and $\to$ is defined through the rewrite rule $r_1 := m \lfloor m12 \rfloor \lfloor m23 \rfloor \to 2$. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical data of $\mathcal{T}$ are $(r_1, 1, r_1)$ and $(r_1, 2, r_1)$.

The critical term associated with the first critical data is

$$\mathfrak{t} := m \lfloor m \lfloor m 1^{(1)} 2^{(1)} \rfloor \lfloor m 2^{(1)} 3^{(1)} \rfloor \rfloor \lfloor m \lfloor m 2^{(1)} 3^{(1)} \rfloor 3 \rfloor$$

and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where

$$\mathfrak{s}_1 := m 2^{(1)} 3^{(1)} \quad \text{and} \quad \mathfrak{s}_2 := m 2^{(1)} \lfloor m \lfloor m 2^{(1)} 3^{(1)} \rfloor 3 \rfloor.$$

These two $\mathcal{S}, \mathrm{C} \cdot \mathbb{N}$-terms are normal forms of $\mathrm{C} \cdot \mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_2 := (\mathfrak{s}_2, \mathfrak{s}_1)$, in this order, since $\mathfrak{s}_2$ contains $\mathfrak{s}_1$ as a subterm.

Similarly, the second critical data makes $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ to add to the set $\{r_1, r_2\}$ the new rewrite rule

$$r_3 := \left(m \lfloor m 1 \lfloor m 1^{(1)} 2^{(1)} \rfloor \rfloor 2^{(1)}, m 1^{(1)} 2^{(1)}\right).$$

We can check (with some further work) that $\{r_1, r_2, r_3\}$ is a normal form of $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$. Therefore, the TRS $(\mathcal{S}, \mathrm{C} \cdot \mathbb{N}, \{r_1, r_2, r_3\})$ is convergent.

Example

Let the TRS $\mathcal{T} := (\mathcal{S}, \mathbb{N}, \to)$ such that $\mathcal{S}$ is the signature containing the binary constant m, and $\to$ is defined through the rewrite rules $r_1 := m \lfloor m12 \rfloor 3 \to m1 \lfloor m23 \rfloor$ and $r_2 := m11 \to 1$. Let also $\rightsquigarrow$ be any $\mathcal{T}$-compatible reduction relation.

The critical term associated with the critical data $(r_1, 1, r_1)$ of $\mathcal{T}$ is $m \lfloor m1^{(1)}1^{(1)} \rfloor 3$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m1^{(1)} \lfloor m1^{(1)}3 \rfloor$ and $\mathfrak{s}_2 := m1^{(1)}3$. These two $\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}$-terms are normal forms of $\mathrm{C}{\cdot}\mathcal{T}$. Therefore, $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ adds to $\to$ the new rewrite rule $r_3 := (\mathfrak{s}_1, \mathfrak{s}_2)$.

Let the TRS $\mathcal{T}' := (\mathcal{S}, \mathrm{C}{\cdot}\mathbb{N}, \{r_1, r_2, r_3\})$. The critical term associated with the critical data $(r_1, 1, r_3)$ of $\mathcal{T}'$ is $m \lfloor m1^{(2)} \lfloor m1^{(2)}3^{(1)} \rfloor \rfloor 3$ and the critical pair associated with this critical data is $(\mathfrak{s}_1, \mathfrak{s}_2)$ where $\mathfrak{s}_1 := m1^{(2)} \lfloor m \lfloor m1^{(2)}3^{(1)} \rfloor 3 \rfloor$ and $\mathfrak{s}_2 := m \lfloor m1^{(2)}3^{(1)} \rfloor 3$. We have that $\mathfrak{s}'_1 := m1^{(3)} \lfloor m1^{(3)} \lfloor m3^{(1)}3 \rfloor \rfloor$ and $\mathfrak{s}'_2 := m1^{(3)} \lfloor m3^{(1)}3 \rfloor$ are respective normal forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ in $\mathcal{T}'$. Therefore, $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$ adds to $\{r_1, r_2, r_3\}$ the new rewrite rule $r_4 := (\mathfrak{s}'_1, \mathfrak{s}'_2)$.

Exercise ○○○○○

Let us consider the previous TRS $\mathcal{T}$. Show that for any $\mathcal{T}$-compatible reduction relation $\rightsquigarrow$, the set $\to$ admits no normal different from Fail in the ARS $\mathsf{Completion}_{\mathcal{S}, \mathbb{N}, \rightsquigarrow}$. In other words, prove that there is no confluent completion of $\mathcal{T}$.